MATH 289
PROBLEM SET 1: INDUCTION

1. The induction Principle

The following property of the natural numbers is intuitively clear:

Axiom 1. *Every nonempty subset of the set of nonnegative integers* $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\}$ *has a smallest element.*

This axiom can be thought of a instance of the *extremal principle* which will be discussed later.

Suppose that k is an integer and we want to prove that

"*$P(n)$ is true for every positive integer n*",

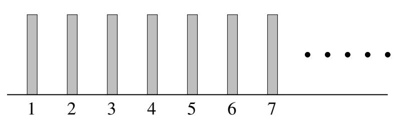
where $P(n)$ is a proposition (statement) which depends on a positive integer n. Proving $P(1)$, $P(2)$, $P(3)$ individually, would take an infinite amount of time. Instead, we can use the *Induction Principle*:

Theorem 2 (Induction Principle)**.** *Assume that k is an integer and $P(n)$ is a proposition for all $n \geq k$.*

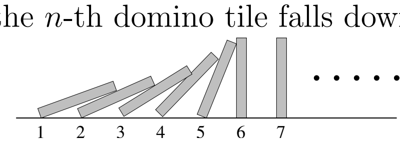
(1) *Suppose that $P(k)$ is true, and*
(2) *for any integer $m \geq k$ for which $P(m)$ is true, $P(m + 1)$ is true.*

Then $P(n)$ is true for all integers $n \geq k$.

For $k = 1$, the induction principle can be compared to an infinite sequence of dominos tiles, numbered 1,2,3, etc.



If the m-th domino tile falls, it will hit the $(m + 1)$-th domino tile and the $(m + 1)$-th domino tile will fall as well. If the first domino tile falls, then *all* domino tiles will fall down. (Here $P(n)$ is the statement: "the n-th domino tile falls down")



Proof of the Induction Principle. Let S to be the set of all nonnegative integers i for which $P(k + i)$ is false. If S is nonempty, then S has a smallest element, say l by Axiom 1. By property (1)), $P(k)$ is true and $l > 0$. The statement $P(k + l - 1)$ is true because l is the minimal element of S. Therefore, statement $P(k + l)$ is true because of property (2). Hence $l \notin S$. Contradiction! We conclude that S is empty, and $P(n)$ is true for all $n \geq k$. □

A typical example of the induction principle is the following:

Example 3. Prove that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \tag{1}$$

for every positive integer n.

Proof. We prove (1) by induction on n. For $n = 1$ we check that

$$1 = \frac{1 \cdot (1+1)}{2}.$$

Suppose that (1) is true for $n = m$. Then

$$1 + 2 + \cdots + m + (m+1) = (1 + 2 + \cdots + m) + (m+1) =$$
$$= \frac{m(m+1)}{2} + (m+1) = \frac{(m+1)(m+2)}{2}.$$

So (1) is true for $n = m + 1$. Now (1) is true for all positive integers n by the induction principle. □

Remark 4. When the German mathematician Carl Friedrich Gauss (1777–1855) was 10 years old, his school teacher gave the class an assignment to add all the numbers from 1 to 100. Gauss gave the answer almost immediately: 5050. This is how (we think) he did it: Write the numbers from 1 to 100 from left to right. Write under that the numbers from 1 to 100 in reverse order.

$$\begin{array}{ccccc} 1 & 2 & 3 & \cdots & 100 \\ 100 & 99 & 98 & \cdots & 1 \\ \hline \underbrace{101 & 101 & 101 & \cdots & 101}_{100} \end{array}$$

Each of the 100 column sums is 101. This shows that

$$2 \cdot (1 + 2 + \cdots + 100) = 100 \cdot 101$$

and

$$1 + 2 + \cdots + 100 = \frac{100 \cdot 101}{2} = 50 \cdot 101 = 5050.$$

This easily generalizes to a proof of (1).

A formula similar to (1) exists for the sums of squares, namely

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2}$$

Example 5. Give and prove a formula for

$$1^3 + 2^3 + \cdots + n^3$$

We have seen similar examples, namely (1) and (2). We can also add the formula

$$1^0 + 2^0 + 3^0 + \cdots + n^0 = n.$$

Let

$$p_k(n) = 1^k + 2^k + 3^k + \cdots + n^k$$

where $k \in \mathbb{Z}_{\geq 0}$. The examples so far suggest that $p_k(n)$ is a polynomial of degree $k+1$ (and that the leading coefficient is $\frac{1}{k+1}$). Let us *assume* that $p_3(n)$ is a polynomial of degree 4.

Since $p_3(0)$ is an empty sum, we have that $p_3(0) = 0$. We can write $p_3(n) = an^4 + bn^3 + cn^2 + dn$. for certain real numbers a, b, c, d. We have

(3)
$$n^3 = p_3(n) - p_3(n-1) = a(n^4 - (n-1)^4) + b(n^3 - (n-1)^3) + c(n^2 - (n-1)^2) + d(n - (n-1)) =$$
$$= a(4n^3 - 6n^2 + 4n - 1) + b(3n^2 - 3n + 1) + c(2n - 1) + d =$$
$$= n^3(4a) + n^2(-6a + 3b) + n(4a - 3b + 2c) + (-a + b - c + d)$$

Comparing coefficients in (3) gives us the linear equations:

(4) $$1 = 4a$$
(5) $$0 = -6a + 3b$$
(6) $$0 = 4a - 3b + 2c$$
(7) $$0 = -a + b - c + d$$

We solve the system of equations and find $a = \frac{1}{4}$, $b = \frac{1}{2}$, $c = \frac{1}{4}$ and $d = 0$. We now should conjecture the following formula:

$$1^3 + 2^3 + \cdots + n^3 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2.$$

Finding this formula was the hard part. It is now not so hard to prove this formula by induction:

Proof. We will prove that

(8) $$1^3 + 2^3 + \cdots + n^3 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2.$$

by induction on n. The case $n = 0$ is clear, because both sides of the equation are equal to 0. If (8) is true for $n = m - 1$, then

$$1^3 + 2^3 + \cdots + (m-1)^3 = \tfrac{1}{4}(m-1)^4 + \tfrac{1}{2}(m-1)^3 + \tfrac{1}{4}(m-1)^2.$$

From this follows that

$$1^3 + 2^3 + \cdots + (m-1)^3 + m^3 = \tfrac{1}{4}(m-1)^4 + \tfrac{1}{2}(m-1)^3 + \tfrac{1}{4}(m-1)^2 + m^3 =$$
$$= \tfrac{1}{4}(m^4 - 4m^3 + 6m^2 - 4m + 1) + \tfrac{1}{2}(m^3 - 3m^2 + 3m - 1) + \tfrac{1}{4}(m^2 - 2m + 1) + m^3 =$$
$$= \tfrac{1}{4}m^4 + \tfrac{1}{2}m^3 + \tfrac{1}{4}m^2,$$

so (8) is true for $n = m$. By induction follows that (8) is true for all $n \in \mathbb{Z}_{\geq 0}$. □

Notice that

$$\tfrac{1}{4}m^4 + \tfrac{1}{2}m^3 + \tfrac{1}{4}m^2 = (\tfrac{1}{2}n(n+1))^2$$

which leads to the following aesthetic formula:

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

Example 6. What is the value of

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots ?$$

Let us compute the partial sums. Perhaps we will find a pattern.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{2}{6}+\frac{4}{6}=\frac{2}{3},$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{2}{3}+\frac{1}{12}=\frac{8}{12}+\frac{1}{12}=\frac{9}{12}=\frac{3}{4},$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}=\frac{3}{4}+\frac{1}{20}=\frac{15}{20}+\frac{1}{20}=\frac{16}{20}=\frac{4}{5}.$$

A pattern emerges. Namely, it seems that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}. \tag{9}$$

Proof. By induction on n we prove:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}. \tag{10}$$

For $n=1$ we check

$$\frac{1}{1\cdot 2}=1-\frac{1}{2}.$$

If (10) is true for $n=m$, then

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{m(m+1)}+\frac{1}{(m+1)(m+2)}=$$

$$=\big(1-\frac{1}{m+1}\big)+\big(\frac{1}{m+1}-\frac{1}{m+2}\big)=1-\frac{1}{m+2}.$$

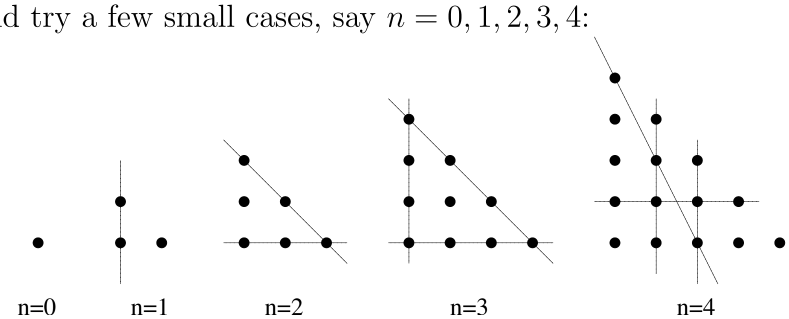
Hence (10) is true for $n=m+1$. By induction, (10) is true for all integers $n\geq 1$. We have

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots=\lim_{n\to\infty}\big(1-\frac{1}{n+1}\big)=1.$$

□

Example 7 (UMUMC, 1988)**.** Let S_n be the set of all pairs (x,y) with integral coordinates such that $x\geq 0$, $y\geq 0$ and $x+y\leq n$. Show that S_n cannot be covered by the union of n straight lines.

First we should try a few small cases, say $n=0,1,2,3,4$:



Notice that S_n is a subset of S_{n+1}. This will be helpful for our induction proof:

Proof. We prove the statement by induction on n, the case $n = 0$ being trivial. Suppose that one needs at least $n + 1$ lines to cover S_n. Define $C_{n+1} = S_{n+1} \setminus S_n$. The set C_{n+1} consists of $n + 2$ points on the line $x + y = n + 1$. Suppose that k lines $\ell_1, \ell_2, \ldots, \ell_k$ cover S_{n+1}.
case 1: One of the lines is equal to the line $x + y = n + 1$. Without loss of generality we may assume that ℓ_k is equal to the line $x + y = n + 1$. Then $\ell_1, \ell_2, \ldots, \ell_{k-1}$ cover S_n because $\ell_k \cap S_n = \emptyset$. From the induction hypothesis follows that $k - 1 \geq n + 1$, so $k \geq n + 2$.
case 2: None of the lines are equal to the line $x + y = n + 1$. Then each of the lines intersects the line $x + y = n + 1$ in at most one point, and therefore it intersects the set C_{n+1} in at most one point. Since C_{n+1} has $n + 2$ elements, there must be at least $n + 2$ lines.

So in both cases we conclude that one needs at least $n + 2$ lines to cover S_{n+1}. □

2. Strong Induction

The following example illustrates that sometimes one has to make a statement stronger in order to be able to prove it by induction.

Example 8. Prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{999,999}{1,000,000} < \frac{1}{1000}.$$

Since $1000 = \sqrt{1,000,000}$ one might suggest that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n}} \tag{11}$$

for all $n \geq 1$. Let us try to prove (11). We can check (11) for small n (which gives some validity to our conjecture that this inequality holds). Suppose that (11) holds for $n = m$:

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2m-1}{2m} < \frac{1}{\sqrt{2m}} \tag{12}$$

We have to prove (11) for $n = m + 1$:

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2m+1}{2m+2} < \frac{1}{\sqrt{2m+2}}. \tag{13}$$

If we divide (13) by (12) we obtain

$$\frac{2m+1}{2m+2} \leq \sqrt{\frac{2m}{2m+2}}. \tag{14}$$

If (12) and (14) are true, then (13) is true. By squaring (14) we see that (14) is equivalent to

$$\left(\frac{2m+1}{2m+2}\right)^2 \leq \frac{2m}{2m+2}$$

and to

$$(2m+1)^2 \leq (2m+2)(2m) \tag{15}$$

So if (15) is true then our induction proof is complete. Unfortunately (15) is not true and we are stuck.

Sometimes it is easier to prove a *stronger* statement by induction:

Proof. We prove

$$\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}} \tag{16}$$

by induction on n. The case $n=1$ is clear because

$$\frac{1}{2} < \frac{1}{\sqrt{3}}.$$

Suppose that (16) is true for $n=m$:

$$\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2m-1}{2m} < \frac{1}{\sqrt{2m+1}} \tag{17}$$

Since

$$(2m+1)(2m+3) = (2m+2)^2 - 1 < (2m+2)^2$$

we have that

$$\left(\frac{2m+1}{2m+2}\right)^2 < \frac{2m+1}{2m+3}$$

and

$$\frac{2m+1}{2m+2} < \sqrt{\frac{2m+1}{2m+3}}. \tag{18}$$

Multiplying (17) by (18) yields

$$\frac{1}{2}\frac{3}{4}\cdots\frac{2m+1}{2m+2} < \frac{1}{\sqrt{2m+3}}, \tag{19}$$

so (16) is true for $n=m+1$. This shows that (16) is true for all positive integers n. In particular, for $n=500,000$ we get

$$\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{999,999}{1,000,000} < \frac{1}{\sqrt{1,000,001}} < \frac{1}{1000}.$$

□

Below is a trickier proof of Example 8.

Proof. Let

$$A = \frac{1\cdot 3\cdot 5\cdots 999,999}{2\cdot 4\cdot 6\cdots 1,000,000}$$

and

$$B = \frac{2\cdot 4\cdot 6\cdots 1,000,000}{3\cdot 5\cdot 7\cdots 1,000,001}.$$

Clearly $A < B$ because

$$\frac{1}{2} < \frac{2}{3}, \frac{3}{4} < \frac{4}{5}, \ldots, \frac{999,999}{1,000,000} < \frac{1,000,000}{1,000,001}.$$

It follows that

$$A^2 < AB = \frac{1}{1,000,001} < \frac{1}{1,000,000}$$

and $A < 1000^{-1}$. □

Example 9. Prove that every integer $n \geq 2$ is a product of prime numbers.

Proof. Let $Q(n)$ be the statement:

"every integer r with $2 \leq r \leq n$ is a product of prime numbers."

We use induction on n to prove that $Q(n)$ holds for all integers $n \geq 2$.

For $n = 2$ the statement is true because 2 is a prime number. Suppose that $Q(m)$ is true. We will prove $Q(m+1)$. Suppose that $2 \leq r \leq m+1$. If $r \leq m$ then r is a product of prime numbers because $Q(m)$ is true. Suppose that $r = m + 1$. If $m + 1$ is a prime number, then $m + 1$ is a product of prime numbers and we are done. Otherwise, $m + 1$ can be written as a product ab with $1 \leq a, b \leq m$. Because $Q(m)$ is true, both a and b are products of prime numbers. Hence $m + 1 = ab$ is a product of prime numbers.

We have shown that $Q(n)$ holds for all $n \geq 2$. In particular, every integer $r \geq 2$ is a product of prime numbers because $Q(r)$ is true. □

3. Induction in definitions

We can also use induction in a definition. For example, the Fibonacci numbers is a sequence of numbers $F_0, F_1, F_2, \ldots$ defined by $F_0 = 0$, $F_1 = 1$ and

$$F_{n+1} = F_n + F_{n-1}, \quad n \geq 1.$$

By (strong) induction on n we can prove that F_n is well-defined for all integers $n \geq 0$. The first few Fibonacci numbers are:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

The sum notation is an example of a recursive definition. Suppose that $f(n)$ is some function. If a, b are integers and $a \leq b + 1$ then we define

$$\sum_{n=a}^{b} f(n)$$

as follows.

$$\sum_{n=a}^{a-1} f(n) = 0$$

and

$$\sum_{n=a}^{b} f(n) = f(b) + \sum_{n=a}^{b-1} f(n) \tag{20}$$

if $b \geq a$.

One can then formally prove by induction that

$$\sum_{n=a}^{c} f(n) = \sum_{n=a}^{b} f(n) + \sum_{n=b+1}^{c} f(n).$$

if $a, b, c \in \mathbb{Z}$ and $a - 1 \leq b \leq c$. (Induction on c. Start with $c = b$.)

Similarly we have the product notation.

$$\prod_{n=a}^{a-1} f(n) = 1$$

and

$$\prod_{n=a}^{b} f(n) = f(b) \prod_{n=a}^{b-1} f(n).$$

if $b \geq a$.

For nonnegative integers m and n with $m \leq n$ we define a binomial coefficient by

$$\binom{n}{m} = \begin{cases} 1 & \text{if } m = 0 \text{ or } m = n \\ \binom{n-1}{m-1} + \binom{n-1}{m} & \text{if } 0 < m < n \end{cases}.$$

If we arrange the binomial coefficients in a triangular shape, we get Pascal's triangle:

$$\begin{array}{ccccccccc} & & & & \binom{0}{0} & & & & \\ & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

After evaluating the binomial coefficients we get:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

4. Exercises

Exercise 1. * Prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n.

Exercise 2. * Show that

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

for all $n \in \mathbb{N}$.

Exercise 3. * Prove that

$$\sum_{i=0}^{n} \binom{m+i}{m} = \binom{m+n+1}{m+1}.$$

for all nonnegative integers m and n.

Exercise 4. * Prove that

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n.$$

Exercise 5. *

(a) Prove that

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

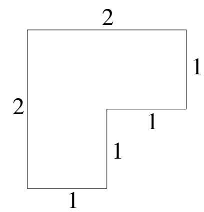
for every real number x and every positive integer n.

(b) If x is a real number with $|x| < 1$ then

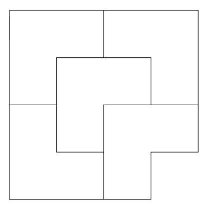
$$1 + x + x^2 + \cdots = \frac{1}{1 - x}.$$

Exercise 6. ** Show that the sum of the squares of two consecutive Fibonacci numbers is again a Fibonacci number.

Exercise 7. ** Cut out a 1×1 corner of a $2^n \times 2^n$ chess board ($n \geq 1$). Show that the remainder of the chess board can be covered with L-shaped tiles (see picture).



The case $n = 2$ is shown below.



Exercise 8. ** Find and prove a formula for

$$1^4 + 2^4 + \cdots + n^4.$$

Exercise 9. ** Show that

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

as follows: Define

$$f(x) = (1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n.$$

and consider $f^{(k)}(0)$ ($f^{(k)}(x)$ is the k-th derivative of $f(x)$).

Exercise 10. ** Define a sequence $a_1, a_2, \ldots$ by $a_1 = \frac{5}{2}$ and $a_{n+1} = a_n^2 - 2$ for $n \geq 1$. Give an explicit formula for a_n and prove it.

Exercise 11 (Division with remainder)**.** ** Suppose that n is a nonnegative integer, and m is a positive integer. Prove that there exist integers q and r with $n = qm + r$ and $0 \leq r < m$.

Exercise 12. ** Give and prove a formula for

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1) \cdot (n+2)}.$$

Exercise 13 (Expansion in base b)**.** *** Suppose that n is a positive integer, and b is an integer ≥ 2. Show that there exist an nonnegative integer m, and integers $a_0, a_1, \ldots, a_m \in \{0, 1, 2, \ldots, b-1\}$ such that

$$n = a_m b^m + a_{m-1} b^{m-1} + \cdots + a_0 \tag{21}$$

and $a_m \neq 0$. Moreover, show that m and $a_0, a_1, \ldots, a_m$ are uniquely determined by n. (We will write $(a_m a_{m-1} \cdots a_0)_b$ for the right-hand side in (21)).

Exercise 14 (Zeckendorf's Theorem)**.** *** Suppose that n is a positive integer. Show that we can write

$$n = F_{i_1} + F_{i_2} + \cdots + F_{i_k}$$

where k is a positive integer, $i_1 \geq 2$ and $i_j \geq i_{j-1} + 2$ for $j = 2, 3, \ldots, k$. Also show that k and $i_1, \ldots, i_k$ are uniquely determined by n.

Exercise 15 (Putnam 1985, B2)**.** *** Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx}(f_{n+1}(x)) = (n+1) f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

Exercise 16 (Putnam 1987, B2)**.** *** Let r, s and t be integers with $0 \leq r$, $0 \leq s$ and $r + s \leq t$. Prove that

$$\frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \cdots + \frac{\binom{s}{s}}{\binom{t}{r+s}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}.$$

Exercise 17. **** Let $n = 2^k$. Prove that we can select n integers from any $(2n-1)$ integers such that their sum is divisible by n.

Exercise 18. *** Find the sum of all fractions $1/xy$ such that $\gcd(x, y) = 1$, $x \leq n$, $y \leq n$, $x + y > n$.

Exercise 19. **** Find and prove a formula for

$$\int_0^{\frac{\pi}{2}} \sin^n(x)\, dx.$$

Exercise 20. **** Suppose that $a_1, a_2, \ldots, a_n$ are positive integers such that $a_1 \leq a_2 \leq \cdots\cdots \leq a_n$. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1$$

implies that $a_n < 2^{n!}$.

Exercise 21. *** Generalize example 7: Let $S_{n,d} \subseteq \mathbb{R}^d$ be the set of all integer vectors $(x_1, \ldots, x_d)$ such that $x_i \geq 0$ for all i and $x_1 + x_2 + \cdots + x_d \leq n$. Show that $S_{n,d}$ cannot be covered with n hyperplanes in $\mathbb{R}^d$.