Lecture Notes - Math 101

Set Theory

Definition *A **Set** is a collection of objects or items called the **elements** or **members** of the set. For a given set, each object or item is either in the set or it isn't (i.e. it can't be in the set more than once).*

Notation *The expression*

$$\{x_1, x_2, \ldots, x_n\}$$

represents the set whose elements are $x_1, x_2, \ldots, x_n$. An item is in this set if and only if it is either x_1 or x_2 or … x_n.

Notation *If A is a set and x is an object then*

$$x \in A$$

means "x is an element of A".

Definition *A **statement** is an expression which is either true or false (but not both).*

Note: $x \in A$ is a statement!

Definition *The **union** of two sets is the set which contains all of the elements of either set and nothing else, i.e. if A and B are sets then the union of A and B is the set of all elements x such that $x \in A$ or $x \in B$. The union of A and B is written $A \cup B$.*

Iteration

Definition *Given the following:*
1. *A **process** that has an input and an output such that any output can be used as an input,*
2. *An initial input called the **seed** (or 0^{th} iteration),*

*to **iterate** the process is to determine the sequence*

$$\text{seed} \to 1^{st} \text{ iteration} \to 2^{nd} \text{ iteration} \to 3^{rd} \text{ iteration} \to \cdots \text{ etc}$$

where the n^{th} iteration is the output of the process when the $(n-1)^{st}$ iteration is input to the process.

Discrete Dynamical Systems

Definition *A **function** is a rule which assigns to each element of a set (called the **domain** of the function) a single element of a set (called the **codomain** of the function). If x is an element of the domain of a function f then $f(x)$ is the element of the codomain of f that f assigns to x.*

Note: If the domain and codomain are the same set then $f(x)$ is the output when x is input to f. Hence, a function from a set to itself is an example of a process we can iterate.

Definition *A function from a set to itself is called a **set theoretic discrete dynamical system** (or simply a **dynamical system** for short in this course).*

Definition *To **iterate** a discrete dynamical system means to compute the sequence*

$$x_0, x_1, x_2, \ldots,$$

where x_0 is the seed (an element of the domain of f) and $x_i = f(x_{i-1})$ for $i > 0$ (i.e. each term is *what the function f gives you as output when the previous term is input to the function). This sequence is called the* ***f-orbit of*** *x_0 (or simply the* ***orbit*** *of x_0). The term x_i is called the i^{th} iteration.*

Definition *The f-orbit of x_0 is a* ***cycle****, if some iteration of the seed is equal to the seed (i.e. if $x_i = x_0$ for some $i > 0$). If the k^{th} iteration equals the seed then the orbit is called a* ***k-cycle****. The seed of a 1-cycle is called a* ***fixed point****. The seed of a k-cycle is called a* ***cyclic point of order*** *k.*

Notation *An overbar indicates repeated terms, i.e.*

$$\overline{x_0, x_1, \ldots, x_n} = x_0, x_1, \ldots, x_n, x_0, x_1, \ldots, x_n, x_0, x_1, \ldots, x_n, \ldots$$

Definition *The f-orbit of x_0 is* ***eventually cyclic****, if some iteration is equal to some prior iteration (i.e. if $x_i = x_j$ for some $i > j$).*

Definition *The* ***minimum period*** *of a cycle is the smallest k such that the k^{th} iteration is equal to the seed. The seed of a cyclic orbit is said to be cyclic, and the seed of an eventually cyclic orbit is said to be eventually cyclic.*

Affine Maps and IFS

Definition *An* ***affine map*** *(or* ***affine transformation****) is a function which sends each point in the plane to another point in the plane. Each affine map is specified by six numbers, r, s, θ, ϕ, e, f. For each choice of six numbers, we name the corresponding affine map $Affine(r, s, \theta, \phi, e, f)$.. The effect of the affine map on a geometric figure is as follows:*

r	*scales the figure horizontally by a factor of $\lvert r\rvert$* *(if r is negative, it also reflects the figure across the y −axis)*
s	*scales the figure vertically by a factor of $\lvert s\rvert$* *(if s is negative, it also reflects the figure across the x −axis)*
θ	*rotates horizontal lines by θ degrees CCW about the point where they intersect the y −axis*
ϕ	*rotates vertical lines by ϕ degrees CCW about the point where they intersect the x −axis*
e	*translates the figure horizontally by an amount e*
f	*translates the figure vertically by an amount f*

Note that if $\theta = \phi$, then the effect of both numbers combined is to rotate the entire figure about the origin by an angle θ counterclockwise (CCW). Negative angles rotate clockwise (CW) instead of counterclockwise. Also note that $Affine(r, s, \theta, \phi, e, f)$ always sends the origin, $(0, 0)$, to the point (e, f).

Formally speaking, $Affine(r, s, \theta, \phi, e, f)$ sends any point (x, y) to the point $(r\cos(\theta)x - s\sin(\phi)y + e, r\sin(\theta)x + s\cos(\phi)y + e)$

Definition *An affine map is a* ***contraction*** *map if it moves every pair of points in the plane closer together.*

Remark *If $\lvert r\rvert < 1$ and $\lvert s\rvert < 1$ then $Affine(r, s, \theta, \phi, e, f)$ is a contraction map.*

Definition *An **Iterated Function System (IFS)**, is a finite set of affine contraction maps.*

The Deterministic Method

Definition *Let $W = \{T_0, T_1, \ldots, T_k\}$ be an IFS with affine contraction maps $T_0, T_1, \ldots, T_k$. Define the following iterative process. Given any closed bounded shape S in the plane, compute the image of S for each of the maps $T_0, T_1, \ldots, T_k$ and collage these images together to form the new image. Iterating this process for a given closed bounded starting shape is called the **Deterministic Method** for producing a fractal from the IFS W. i.e. given a seed shape A_0 we define the function $W(A) = T_0(A) \cup T_1(A) \cup \cdots \cup T_k(A)$ and compute the W-orbit of A_0.*

Important Facts about IFS's and the Deterministic Method and IFS's:

1. Iterating the IFS by the deterministic method will produce images which converge to a **unique** image that is associated with the IFS. This image is called the **attractor** of the IFS. This image is usually a fractal. We say that the attractor of an IFS is the fractal associated with the IFS (or produced by the IFS).
2. The Deterministic Method will produce the **same** attractor for W, no matter WHAT seed we start with!
3. All HeeBGB fractals can be produced by an IFS (they are just a special kind of IFS).
4. The fractal that is produced by an IFS is the **only** (closed bounded) shape that the IFS maps to itself! Thus the attractor is a fixed point for the iterative process associated with W, i.e. the attractor A is the only (closed bounded) shape such that $W(A) = A$. So the attractor of an IFS is composed of a finite number of affine copies of itself.

Addresses

Definition *Let $W = \{T_0, T_1, \ldots, T_k\}$ be an IFS with affine contraction maps $T_0, T_1, \ldots, T_k$, let A be the attractor of W, and let p be any point in A. We say p has **address** $s_0 s_1 s_2 s_3 \cdots$ if $p \in T_{s_0} T_{s_1} T_{s_2} \cdots T_{s_n}(A)$ for every n, that is to say p is in the image of the attractor under T_{s_0} and it is also in the image of the attractor under T_{s_1} followed by T_{s_0} and it is also under the image of the attractor under T_{s_2} followed by T_{s_1} followed by T_{s_0} and so on.*

Facts about addresses:

1. A point can have more than one address.
2. Every sequence of numbers between 0 and k inclusive is the address of a unique point in the attractor.
3. If two points have addresses which have match for a long initial segment, then the points will be close together and the more terms in the address that match, the closer together the points will be.

Random methods

Definition *The **Chaos Game** is the following iterative process. Choose points in the plane $P_1, P_2, \ldots, P_k$ called the **goal** points. For the seed, choose any point q in the plane and plot it. The process is*

1. *The input is the current point q.*
2. *Randomly select one of the goal points, P_i.*
3. *Move halfway between q and the goal point you selected, i.e. find the midpoint, m, of the line segment connecting q and P_i.*

4. *Plot point m.*
5. *Output the point m (which becomes the new input q for the next iteration).*

Facts about the Chaos Game:

1. Regardless of the starting point you select, iterating this process will produce an image that converges to a single attractor which is only dependent on the choice of goal points.
2. The image that it produces is the attractor of an IFS whose affine maps have the effect of moving every point in the plane half way towards one of the goal points.

Definition *Let $W = \{T_0, T_1, \ldots, T_k\}$ be an IFS with affine contraction maps $T_0, T_1, \ldots, T_k$. Start with any point q in the plane as a seed and plot it. Then plot the orbit of q by iterating the following process:*
1. *The input is the current point q.*
2. *Randomly select one of the affine maps, T_i.*
3. *Compute $T_i(q)$ and call this point m, i.e. m is the image of q under the affine map selected in step #2.*
4. *Plot point m.*
5. *Output the point m (which becomes the new input q for the next iteration).*

*This method is called the **random iteration method** for producing the attractor of the IFS W.*

Facts about the Random Iteration Method:

1. The random iteration method will produce the attractor of the IFS W as long as every possible finite subsequence eventually appears.
2. The reason it works is because of the following facts:
   a. The image of any point in the plane under one of the affine maps is closer to the attractor than the original point, and after several iterations the point will be approximately in the attractor (within the accuracy of the plot).
   b. Once a point is in the attractor, it must stay in the attractor because the attractor is the unique closed bounded shape that the IFS maps to itself.
   c. Any point p in the attractor has an address. If we apply the first k affine maps specified by the address to any point q in the attractor we will obtain a point that is very close to p (and the larger k is the closer to p the image of will be.
   d. Every finite sequence of digits eventually appears in a typical random sequence, so that every point in the attractor is eventually plotted (within the accuracy of our plots).

Fractal Data Analysis

Definition *Given a set of data values, $d_1, d_2, d_3, \ldots$ we can test the data for randomness as follows.*
1. *Group the data into four categories, A, B, C, and D.*
2. *Play the Chaos Game with the goal points $(0,0), (1,0), (0,1)$, and $(1,1)$ corresponding to categories A, B, C, and D respectively, but using your (categorized) data sequence as the source of "random" numbers. (Equivalently, use the random iteration method to plot the attractor of $HeeBGB(Up, Up, Up, Up)$ with the (categorized) data as your source of "random" numbers.)*

*This technique is called **Fractal Data Analysis.***

Facts about fractal data analysis:

1. If your data is truly random, then the image produced should be the attractor of $HeeBGB(Up, Up, Up, Up)$, i.e. a solid square. If your data is not random, some subsequences

will be missing and will cause holes in the square (in a fractal manner) by avoiding all points with addresses that contain the missing subsequences.

2. It is usually only effective if you have a lot of data... usually at least 200 points or more before it is even moderately effective. The more data you have the more effective the test is at analyzing it.

Exponents and Logarithms

Definition *If a is any number and n is a positive whole number then*

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

Also

$$a^0 = 1$$

and

$$a^{-n} = \frac{1}{a^n}.$$

Facts about exponents:

1. For any number a,

$$a^1 = a$$

2. For any nonzero numbers $a, b,$ and any positive whole number n

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

3. For any nonzero numbers $a, b,$ and any positive whole number n

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

4. It is possible to define a^n for exponents n which are any positive number, not just for positive whole numbers, in such a way that the above properties still hold. In this case, if n is between whole numbers m and $m+1,$ then $a^m < a^n < a^{m+1}$.

Definition *If a is a positive number other than 1 and x is any positive number then*

$$log_a(x)$$

is the power you have to raise a to in order to get x. i.e.

$$a^{log_a(x)} = x$$

and

$$log_a(a^x) = x.$$

Definition $log(x)$ *means* $log_{10}(x)$.

Properties of logarithms:

1. If a is a positive number other than one and b, c are positive numbers then
   a. $\log_a(a^b) = b$
   b. $a^{\log_a(b)} = b$
   c. $\log_a(1) = 0$
   d. $\log_a(a) = 1$
   e. $\log_a(bc) = \log_a(b) + \log_a(c)$

f. $\log_a(\frac{b}{c}) = \log_a(b) - \log_a(c)$
g. $\log_a(b^c) = c\log_a(b)$

Similarity Dimension

Definition *An affine map $Affine(r,s,\theta,\phi,e,f)$ is called a **similarity** if $|r| = |s|$ and $\theta = \phi$.*

Definition *A shape is **self similar** if it is the union of a finite number of strictly smaller similar copies of itself. If all of the copies are the same size, then the shape is called **strictly self-similar**.*

Comments:

- The attractor of any HeeBGB IFS is strictly self-similar.
- The attractor of any IFS whose affine maps are all similarities is self-similar.
- The attractor of any IFS whose affine maps are all similarities and whose scaling factors r, s all have the same absolute value is strictly self-similar.

Definition *If A is a strictly self-similar shape which consists of N copies of itself with scaling factor s (which don't overlap too much) then the similarity dimension of A is the unique number d such that*

$$N = \left(\frac{1}{s}\right)^d.$$

Note: If we solve this equation for d we obtain the formula:

$$d = \frac{\log(N)}{\log(\frac{1}{s})}$$

Grid Dimension

Definition *Given a shape A, if we cover A with two uniform grids of grid width w_1 and w_2 respectively, and if the number of grid squares that contain a piece of A is N_1 for the first grid and N_2 for the second grid then the **grid dimension** of A is approximately*

$$d = \frac{log(N_2) - log(N_1)}{log(w_1) - log(w_2)}.$$

Chaos

Definition *A set of points A is **dense** in a set of points B if every point of B is arbitrarily close to a point in set A ("arbitrarily close" means "within any positive distance no matter how small")*

Definition *Let f be a function from a set of points X to itself. We say that f has **sensitive dependence on initial conditions** if a slight change in the seed eventually causes a significant difference in the orbits, i.e. if slightly different seeds have significantly different orbits after enough iterations.*

Definition *Let f be a function from a set of points X to itself. We say that f is **transitive** if there is an orbit of f which is dense in X.*

Definition *(Devaney's Definition of Chaos) Let f be a function from a set of points X to itself. We say f is **chaotic** if*

1. *f has dense periodic points.*

2. *f is transitive.*

3. *f has sensitive dependence on initial conditions.*

Facts about chaotic maps:

- The function $L(x) = 4x(1-x)$ is chaotic on the set of numbers between 0 and 1 inclusive. $L(x)$ is called the **logistic map**.
- The function $Q(x) = x^2 - 2$ is chaotic on the set of numbers between -2 and 2 inclusive. $Q(x)$ is called the **Quadratic map**.

The Shift Map

Definition *Let $W = \{T_0, T_1, \ldots, T_n\}$ be a totally disconnected IFS (each point has a unique address) with attractor A. For any address $s_0s_1s_2\ldots$ define $\Phi(s_0s_1s_2\ldots)$ to be the point in the attractor that has address $s_0s_1s_2\ldots$. Define a function S from the attractor to itself by $S(\Phi(s_0s_1s_2\ldots)) = \Phi(s_1s_2\ldots)$, i.e. the shift map sends the point whose address is $s_0s_1s_2\ldots$ to the point whose address is $s_1s_2\ldots$.*

Facts about the Shift Map:

- **The shift map is chaotic**!!!

Complex Numbers

Definition *The set of complex numbers, $\mathbb{C}$, consists of all expressions of the form $a + bi$ where a and b are real numbers and i is a number which has the property that $i^2 = -1$. The real number a is called the **real part** of $a + bi$ and the real number b is called the **imaginary part** of $a + bi$. The expression $a + bi$ is called the **standard form** for a complex number.*

Abbreviations: We write

a as an abbreviation for $a + 0i$

bi as an abbreviation for $0 + bi$

$a + i$ as an abbreviation for $a + 1i$

Both of these abbreviation are acceptable standard forms for complex numbers.

Definition *Let $a + bi$ and $c + di$ be complex numbers. Then*

$$(a + bi) + (c + di) = (a + b) + (c + d)i$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Comment on Complex Multiplication: We usually do not memorize the above formula but rather use the distributive law and the fact that $i^2 = -1$ to simplify the product into standard form:

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= ac + (ad + bc)i + bd(-1) \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

Definition *Let $a + bi$ be a complex number. The ordered pair (a, b) is called the **ordered pair associated with** $a + bi$. Similarly we say $a + bi$ is the **complex number associated with** (a, b). This allows us to plot complex numbers on a Cartesian coordinate plane by plotting their associated ordered pair, and also to label the points on the Cartesian coordinate plane with* *complex numbers. When the points in the plane are labeled with complex numbers, we call this the **Complex Plane**. The x-axis of the complex plane is called the **real axis** and the y-axis is called the **imaginary axis.***

Definition *Let $a + bi$ be a complex number. Define*

$$|a + bi| = \sqrt{a^2 + b^2}.$$

Comments on absolute value: The absolute value of a complex number measures the distance the number is from the zero (the origin) on the complex plane.

Mandelbrot and Julia Sets

Definition *Let f be a function which takes complex numbers as input and outputs complex numbers. The f-orbit of seed z is **bounded**, if there is a positive real number N such that every term of the orbit has absolute value less than N.*

Definition *Let c be a complex number. Define $Q_c(z) = z^2 + c$.*

Definition *Let c be a complex number. The **filled in Julia set** associated with c, K_c, is the set of all complex numbers w such that the Q_c-orbit of w is bounded. The boundary of K_c is called the **Julia set** associated with c and is denoted J_c.*

Definition *The **Mandelbrot Set**, M, is the set of all complex numbers c, such that the Q_c-orbit of 0 is bounded.*

Facts about Mandelbrot and Julia Sets:

- If any term of the Q_c-orbit of w has absolute value greater than 2, then the orbit is not bounded.
- The complex number c is in the Mandelbrot set if and only if J_c is connected.