1. Math 210 Finite Mathematics

- Chapter 5.2 and 4.3
- Annuities
  Mortgages
  Amortization
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- Northern Illinois University
- Math 210 Website: http://math.niu.edu/courses/math210

2. Math 210 FINAL EXAM

Thursday, December 10, 2009
Noon - 1:50 pm
Faraday 143

Review
Renata: Mon Dec 7 6–8 pm DU 176
Michelle: Tues Dec 8 12–2 pm DU 204
Office Hours
Richard: Wed Dec 9 2–4 pm Watson 344

3. Saving Money Every Month

- Suppose you put M dollars in the bank every month, for six months.
- If the bank pays 0.005 monthly interest (6% annual interest), your money grows like this:

month	formula	calculated
1	$1.005^5 M$	$1.0253M$
2	$1.005^4 M$	$1.0202M$
3	$1.005^3 M$	$1.0150M$
4	$1.005^2 M$	$1.0100M$
5	$1.005^1 M$	$1.0050M$
6	M	M
Total	Sum	$6.0755M$

4. Final Balance

- After 6 months, you account will be worth

$$1.005^5M + 1.005^4M + 1.005^3M + 1.005^2M + 1.005^1M + M$$

- Factoring out an M, this sum is
- $(1.005^5 + 1.005^4 + 1.005^3 + 1.005^2 + 1.005^1 + 1)M$
- We want a formula for the sum inside the parentheses.

5. A Summation Formula

To compute $S = x^5 + x^4 + x^3 + x^2 + x^1 + 1$

use the following trick:

Multiply S by x then subtract S.

$$\begin{aligned} xS &= x^6 + x^5 + x^4 + x^3 + x^2 + x^1 \\ S &= \qquad x^5 + x^4 + x^3 + x^2 + x^1 + 1 \\ xS - S &= x^6 - 1 \end{aligned}$$

All the other terms on the right cancel!

The rest is easy:

$$xS - S = x^6 - 1 \implies (x-1)S = x^6 - 1 \implies S = \frac{x^6-1}{x-1}$$

6. Using the Summation Formula

We saw that if you deposit M dollars in the bank for six consecutive months, then the balance after six months is

$S = (1.005^5 + 1.005^4 + 1.005^3 + 1.005^2 + 1.005^1 + 1)M$

Plugging $x = 1.005$ into the Summation Formula

$$\boxed{1 + x + x^2 + \cdots + x^5 = \frac{x^6 - 1}{x - 1}}$$

gives

$$S = \frac{1.005^6 - 1}{1.005 - 1} M = \frac{1.005^6 - 1}{.005} M$$

7. Final Calculation

- If you deposit $M = 100$ dollars every month for six months, at a monthly rate of $i = .005$, you will have
- $S = \dfrac{1.005^6 - 1}{.005} 100 = 607.55$ dollars after 6 months.
- The 7.55 represents accumulated monthly interest.

8. Monthly Savings formula

$$\boxed{S = \frac{(1+i)^n - 1}{i} M}$$

where

- $M =$ amount saved per month
- $S =$ ending balance
- $i = r/12 =$ monthly interest rate
- $n =$ number of months

9. Long Term Savings

- Suppose you saved $M = 100$ dollars every month, for 45 years, at a monthly rate of $i = .005$.
- Here
  - $M = 100$
  - $i = .005$
  - $n = 12 \times 45 = 540$
- After 45 years the account balance would be
- $S = \dfrac{1.005^{540} - 1}{.005} 100 = 275,599$ dollars.
- Your monthly contributions were $540 \times 100 = 54,000$, so most of the growth in the account is due to accrued interest.

10. Saving for Retirement

- You and your sister have different ideas about saving for your retirement at age 65.
- At age 25 you start put aside 200 every month into an IRA yielding 7 percent interest.
- After ten years, at age 35, you decide that family obligations require you to save the $200 for college money for your three kids.
- So you stop putting money into the account, but let it continue to collect 7% interest until you retire at age 65.

11. Saving for Retirement Cont'd

- Your sister, on the other hand, waits until her 45th birthday to begin saving for her retirement.
- Like you, she has $200 taken out of her paycheck every month into a 7% IRA account.
- At age 65, who has saved more money: you or your sister?

12. Your Sister

- Let's look at your sister first, since her situation is easier to analyze.
- The monthly interest rate is $i = .07/12$ for $n = 240$ months.
- Since $M = 200$, the monthly savings formula gives an ending balance after 20 years of
- $\frac{(1+.07/12)^{240}-1}{.07/12}200 = 104,185$
- more than doubling the $240 \times 200 = 48,000$ she has invested.

13. Your Turn

- You on the other hand have invested one half as much money as your sister: $120 \times 200 = 24,000$.
- After ten years, when you are 35, your ending balance is computed by the Monthly Savings Formula (with $i = .07/12$ and $n = 120$ months):
- $S = \frac{(1+.07/12)^{120}-1}{.07/12}200 = 34,616.96$

- Now this money earns compound interest over the next 30 years or $30 \times 12 = 360$ months, so that at age 65 your IRA account will be worth
- $(1 + .07/12)^{360} \times S = 280,968.48$
- You made 2.7 times as much as your sister, although you only invested half as much money.

14. Advice

- Moral: How much you accumulate for retirement depends upon three things:
  - (i) when you start saving,
  - (ii) how much you manage to save, and
  - (iii) how much your investments return over the long run.
- Of the three, when you start saving turns out to be the most important.
- Moral: Invest when you are young!

15. Monthly Payment M to obtain Balance S

$$\boxed{M = \frac{i}{(1+i)^n - 1} S}$$

This formula comes from the Monthly Savings Formula

$$\boxed{S = \frac{(1+i)^n - 1}{i} M}$$

16. Saving for College

- Kaylee's parents want to put aside money every month so that their daughter will have $25,000 for college when she turns 18.
- At 5% annual interest, how much do they need to save per month?
- Set the variables:
  - amount saved per month: M
  - desired ending balance: $25,000$
  - monthly interest rate: $\begin{aligned} i &= .05/12 \\ &= .004166667 \end{aligned}$
  - number of months: $12 \times 18 = 216$

- $M = \dfrac{.004166667}{(1.004166667)^{216} - 1} 25,000 = 71.60$

17. Annuities

A sequence of payments made at regular intervals is called an annuity.

Special types of annuities:

- certain annuity — the term is a finite time period
- ordinary annuity — eah payment is made at the end of a payment period
- simple annuity — the payment period coincides with the interest conversion period

The annuities we consider are (i) certain; (ii) ordinary; (iii) simple; and (iv) the periodic payments are all the same size.

18. Future Value of an Annuity

$$\boxed{S = \frac{(1+i)^n - 1}{i} M}$$

where

- M = amount paid per investment period
- S = ending balance
- i = periodic interest rate
- n = number of periods

Note that the interest periods need not be months, though frequently they are.

19. Present Value of an Annuity

The idea is simply to determine what principal P would you need to start with in order to end up with the balance S after n interest periods, compounded per period at an annual rate r?

That is, we set the compund interest formula equal to the future value of an annuity formula:

$$P(1+i)^n = \frac{(1+i)^n - 1}{i} M$$

$$\implies \quad P = \frac{(1+i)^n - 1}{(1+i)^n\, i} M$$

$$\implies \quad \boxed{P = \frac{1-(1+i)^{-n}}{i} M}$$

20. Annuities

- If you ever inherit (or win from the lottery) a large sum of money, you may wish to set up an annuity, typically with an insurance company or financial institution.
- An annuity works exactly like a home mortgage, reversing the roles of lender and borrower.
- With a home mortgage, a financial institution gives you a large sum of money (to buy your house) in exchange for your monthly payments over the next n months.
- With an annuity, you give the financial institution a large sum of money and they agree to pay you the monthly (or annual) payments over a certain time period.

21. The Lottery

- Typically lottery winnings are always paid as annuities these days.
- Suppose you win one million dollars in a contest.
- The rules of the contest state that you will be paid $40,000$ per year over the next 25 years.
- If the interest rate is $r = .065$, how much does the company sponsoring the contest need to pay to set up the annuity?
- **Solution:** Compute the present value of an annuity which pays $40,000$ per year at a rate .065 over 25 years.
- Note that payments are annual, not monthly.

22. Lottery Continued

- Variables in the Present Value Formula

$$\begin{aligned} \text{annual interest rate}: &\quad i = r = .065 \\ \text{number of payments (in years)}: &\quad n = 25 \\ \text{annual payment}: &\quad M = 40,000 \end{aligned}$$

- Present Value Calculation $P = \frac{1-(1.065)^{-25}}{.065} 40000 = 487,915.07$
- So it costs the company less than 488 thousand dollars to give away a million dollars.

23. Amortization of Loans

Suppose you purchase a home or car by borrowing a principal of P dollars.

Then P may be thought of as the present value of an annuity and the value of M in the formula

$$P = \frac{1-(1+i)^{-n}}{i} M$$

is the amount you need to pay every month in order to pay off the loan in n payments.

Solving for M in terms of P:

$$M = \frac{i}{1-(1+i)^{-n}} P$$

24. Monthly Payback formula

$$\boxed{M = \frac{i}{1-(1+i)^{-n}} P}$$

where

- M = monthly payment
- P = principal
- $i = r/12$ = monthly interest rate
- n = number of months

25. Calculator Tips

- You need to use parentheses around $1+i$
- You need parentheses around the entire denominator
- You need to use the negative button (not subtract) on the exponent $-n$
- You can store $i = r/12$ in the memory of your calculator
- The Monthly Payback Equation then becomes

$$\text{RCL MEM} \div \Big(1 - \big(1 + \text{RCL MEM}\big) \wedge -n\Big) \times P$$

26. Amortization

To amortize a loan means to set aside money regularly for future payment of the debt.

The general formula for the amortization of a loan is

$$\boxed{M = \frac{i}{1-(1+i)^{-n}}\,P}$$

where

- M = payment per period
- P = principal debt
- i = periodic interest rate $= r/$number of periods per year
- n = number of periods

Typically, the periods are months.

27. Buying a Car

- You want to buy a Dodge Neon for \$12,500.
- Which is the better deal:
  - \$1000 cash back and a 3 year bank loan at 8.5% interest or
  - the promotional 1.9% loan from Dodge for the entire \$12,500?

28. Option 1. Cash Back

- For the cash back option, you need to borrow
  - principal borrowed : $P = 11,500$
  - annual interest rate : $r = .085$
  - monthly interest rate : $\begin{aligned} i &= r/12 \\ &= .007083333 \end{aligned}$
  - number of months : $n = 12 \times 3 = 36$
- The monthly payment for this option is

  $M = \dfrac{.007083333}{1 - 1.00708333^{-36}} 11500 = 363.03$

29. Option 2. Low Interest

- For the 1.9% interest option, you need to borrow
  - principal borrowed : $P = 12,500$
  - annual interest rate : $r = .019$
  - monthly interest rate : $\begin{aligned} i &= r/12 \\ &= .001583333 \end{aligned}$
  - number of months : $n = 12 \times 3 = 36$
- The monthly payment for this option is
- $M = \dfrac{.001583333}{1 - 1.00158333^{-36}} 12500 = 357.49$

30. Comparison

- Option 1. Payment is \$363.03
- Option 2. Payment is \$357.49
- **Conclusion:** Option 2 (with the lower interest) is slightly better.
- It saves about \$5.54 per month.
- Over 36 months, this savings amounts to almost \$200.

31. Principal in terms of Monthly Payment

$$\boxed{P = \frac{1 - (1+i)^{-n}}{i} M}$$

This is just the formula for the present value of an annuity.

32. Buying a House

- You wish to buy a house on a 30 year fixed mortgage.
- The interest rate is a fixed 8% (per year).
- The largest mortgage you can afford is \$1000 per month.
- What is the largest principal you can afford if the interest rate is
  - 8 percent
  - 4.25 percent

33. Buying a House — High Interest

- Variables:
  - annual interest rate : $r = .08$
  - monthly interest rate : $i = r/12 = .0066666667$
  - number of months : $n = 12 \times 30 = 360$
  - monthly payment : $M = 1000$
- Largest Principal You Can Borrow

$$P = \frac{1 - (1.0066667)^{-360}}{.0066667} 1000 = 136,283.50$$

34. Buying a House — Low Interest

- Variables:
  - annual interest rate : $r = .0425$
  - monthly interest rate : $i = r/12 = .00354166667$
  - number of months : $n = 12 \times 30 = 360$
  - monthly payment : $M = 1000$
- Largest Principal You Can Borrow

$$P = \frac{1 - (1.0066667)^{-360}}{.0066667} 1000 = 203,276.86$$

35. Buying a House Comparison

Interest	Principal
8%	136,283.50
4.25%	203,276.86

Note: the amount that you will pay — one thousand dollars per month for 30 years — is the same for both cases.

36. Interest Paid 30 - year loan

year	principal	interest	payoff
first	91.44	908.56	136283.50
1	99.03	900.97	135145.04
2	107.25	892.75	133912.08
5	136.24	863.76	129564.53
10	202.97	797.03	119554.30
15	302.40	697.60	104640.61
20	450.52	549.48	82421.51
21 4mo	501.06	498.94	74841.26
25	671.21	328.79	49318.48
29	923.36	76.64	11495.84
last	993.38	6.62	993.44

37. Reducing the Interest

- $$\begin{aligned}\text{Total Interest } &= 360M - P \\ &= 360 \times 1000 - 136,283.50 \\ &= 223,716.50\end{aligned}$$
- If you paid the same principal 136,283.50 off in a 15 year loan at the same interest rate 8%, what would the monthly payments be?
- $i = .08/12 \qquad n = 180 \quad P = 136,283.50$
- $$\begin{aligned}M &= \frac{.0066667}{1-(1.0066667)^{-180}} \cdot 136,283.50 \\ &= 1302.40\end{aligned}$$

38. Interest Paid 15-year loan

year	principal	interest	pay off
first	393.84	908.56	136283.50
1	426.53	875.87	131380.18
2	461.93	840.47	126069.89
5	586.77	715.63	107345.13
6 4mo	652.58	649.82	97472.54
10	874.19	428.21	64231.42
14	1202.59	99.81	14970.86
last	1293.78	8.62	1292.44

$$\text{Total Interest } = 180M - P = 180 \times 1000 - 136,283.50 = 98,147.80$$

39. Comparison

- **30 - year loan**

$$\begin{aligned}\text{Monthly Payment :} &\quad M = 1000\\ \text{Number of months :} &\quad n = 360\\ \text{Total paid :} &\quad 360,000\\ \text{Total interest :} &\quad 223,716.50\end{aligned}$$

- **15 - year loan**

$$\begin{aligned}\text{Monthly Payment :} &\quad M = 1302.40\\ \text{Number of months :} &\quad n = 180\\ \text{Total paid :} &\quad 234,432\\ \text{Total interest :} &\quad 98,148\end{aligned}$$

40. Advice

- The 15-year loan saves over $125,000 in interest.
- Moral: Go for the shorter loan if you can afford it.

41. A Calculator Error

- If \$750 is the largest monthly house payment you can afford, what is the largest principle you can borrow from the bank over 15 years at 7.4% interest?
- The Maximum Principle Formula says
- $P = \dfrac{1-\big(1+.074/12\big)^{-180}}{.074/12} \times 750$
- A student from a previous semester typed
- $(1 - (1 + .074 \div 12) \wedge -180)\ \div .074 \div 12 \times 750 =$
- and got the answer \$565.30
- What went wrong?
- Does this answer seem reasonable?

42. Credit Cards Advantages

- convenient
- necessary for auto rental
- phone orders
- records and receipts
- eliminates worry of cash

43. Credit Cards Disadvantages

- usually high interest rate
- encourages excess spending
- balance can keep growing
- large proportion spent on interest
- "a continued life of debt"

44. Credit Card Interest

- You owe \$2000 on your Visa Card
- The interest rate is 18%
- You are required to pay 2% of the balance per month
- The monthly interest is $\dfrac{.18}{12} = .015$

- On a balance of \$2000, you will pay

$$0.015 \times 2000 = 30$$

- Your minimal monthly payment: \$ 40.00

45. Credit Card Interest Cont'd

- The interest you pay: \$ 30.00
- Amount paid on balance: \$ 10.00
- Getting an A in this class: priceless
- Notice that 3/4 of your payment is going directly to interest.
- At this rate, you will pay \$8000 for the \$2000 worth of merchandise you bought.

46. How to get out of credit card debt

- You owe \$2000 to Visa.
- The interest rate is 18% or 1.5% per month.
- You would like to pay this off in two years.
- First, cut up the card with scissors or at least promise not to use it for two years.
- Now use the Monthly Payback Formula:

$$\begin{aligned} \text{principal borrowed} : &\quad P = 2,000 \\ \text{annual interest rate} : &\quad r = .18 \\ \text{monthly interest rate} : &\quad i = r/12 = .015 \\ \text{number of months} : &\quad n = 12 \times 2 = 24 \end{aligned}$$

47. Getting out of credit card debt

- Calculate the monthly payment:
- $M = \dfrac{.015}{1 - 1.015^{-24}} 2000 = 99.85$
- Send Visa a check for \$99.85 every month for two years and you will be out of debt.
- Total Payments: $24 \times 99.85 = 2396.40$
- Total Interest: $2396.40 - 20000 = 396.40$