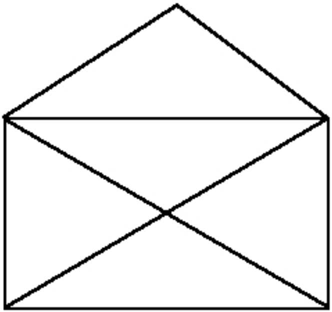
Discrete Math

Instructor: Mike Picollelli

Day 12

And Now, The Theory of Graphs.

Can you draw this picture without lifting your pencil from the paper or repeating lines?



Just A Bunch Of Dots And Lines.

Definition: A **graph** $G = (V, E)$ is a set V of vertices and a set E of edges E, where an edge $e \in E$ is an unordered pair of vertices (a subset of V of size 2).

Just A Bunch Of Dots And Lines.

Definition: A **graph** $G = (V, E)$ is a set V of vertices and a set E of edges E, where an edge $e \in E$ is an unordered pair of vertices (a subset of V of size 2).

If we allow edges to be repeated, we call the resulting object a **multigraph**.

Just A Bunch Of Dots And Lines.

Definition: A **graph** $G = (V, E)$ is a set V of vertices and a set E of edges E, where an edge $e \in E$ is an unordered pair of vertices (a subset of V of size 2).

If we allow edges to be repeated, we call the resulting object a **multigraph**.

For simplicity, if $u, v \in V$ are vertices and E contains the edge $\{u, v\}$, we'll write it as uv, and say that u and v are **adjacent**.

Oh, The Price We Must Pay To Play!

Assume $G = (V, E)$ is a graph we're looking at.

Oh, The Price We Must Pay To Play!

Assume $G = (V, E)$ is a graph we're looking at.

Definitions:

- A **trail** of length is a sequence of vertices $(v_1, v_2, \ldots, v_n)$ so that $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, n-1$, and no edge is repeated.

Oh, The Price We Must Pay To Play!

Assume $G = (V, E)$ is a graph we're looking at.

Definitions:

- A **trail** of length is a sequence of vertices $(v_1, v_2, \ldots, v_n)$ so that $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, n-1$, and no edge is repeated.
- A **path** is a trail in which no vertex is repeated, either.

Oh, The Price We Must Pay To Play!

Assume $G = (V, E)$ is a graph we're looking at.

Definitions:

- A **trail** of length is a sequence of vertices $(v_1, v_2, \ldots, v_n)$ so that $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, n-1$, and no edge is repeated.
- A **path** is a trail in which no vertex is repeated, either.
- A **circuit** is a trail which starts and ends at the same vertex.

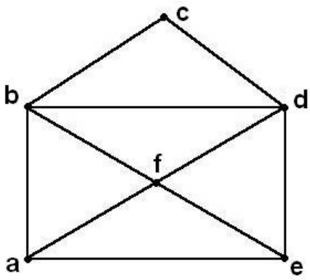
Oh, The Price We Must Pay To Play!

Assume $G = (V, E)$ is a graph we're looking at.

Definitions:

- A **trail** of length is a sequence of vertices $(v_1, v_2, \ldots, v_n)$ so that $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, n-1$, and no edge is repeated.
- A **path** is a trail in which no vertex is repeated, either.
- A **circuit** is a trail which starts and ends at the same vertex.
- A **cycle** is a circuit in which the only repeated vertex is the first one *and* is on at least 3 vertices.

Back To The Fut-Past.



We can view this as a graph G with vertex set

$$V = \{a, b, c, d, e, f\}$$

and edge set

$$E = \{ab, af, ae, bc, bd, bf, cd, de, df, ef\}.$$

Curse You, Euler!

Definition: An **Eulerian trail** is a trail that includes every edge of G. An **Eulerian circuit** is a circuit that includes every edge.

Curse You, Euler!

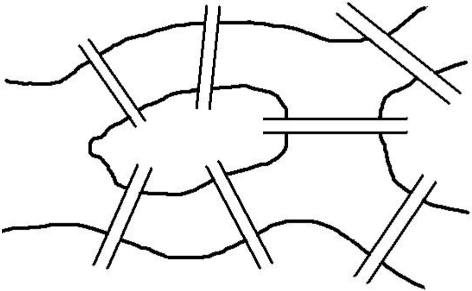
Definition: An **Eulerian trail** is a trail that includes every edge of G. An **Eulerian circuit** is a circuit that includes every edge.

Thus, our previous problem reduces to finding an Eulerian trail: one such solution is

$$(a, b, c, d, e, a, f, d, b, f, e).$$

Curse You, Euler!

The Bridges of Königsberg:



Starting on a bank, can you traverse each edge exactly once, returning to the bank you started on?

Euler Gets Too Much Credit.

<u>**Definition:**</u> The **degree** of a vertex $v \in V$, denoted $d(v)$, is the number of edges in E that contain v.

Euler Gets Too Much Credit.

Definition: The **degree** of a vertex $v \in V$, denoted $d(v)$, is the number of edges in E that contain v.

Definition: A graph is **connected** if for any distinct $u, v \in V$, there is a path between u and v.

Euler Gets Too Much Credit.

<u>Definition:</u> The **degree** of a vertex $v \in V$, denoted $d(v)$, is the number of edges in E that contain v.

<u>Definition:</u> A graph is **connected** if for any distinct $u, v \in V$, there is a path between u and v.

<u>Theorem:</u> **(Euler, 1735)** A connected (multi)graph G has an Eulerian circuit if and only if every vertex has even degree.

Euler Gets Too Much Credit.

Definition: The **degree** of a vertex $v \in V$, denoted $d(v)$, is the number of edges in E that contain v.

Definition: A graph is **connected** if for any distinct $u, v \in V$, there is a path between u and v.

Theorem: **(Euler, 1735)** A connected (multi)graph G has an Eulerian circuit if and only if every vertex has even degree.

Theorem: **(Euler, 1735)** A connected (multi)graph G has an Eulerian trail if and only if it has exactly two vertices of odd degree.

Another Result of Euler's.

The Handshaking Lemma: Let $G = (V, E)$ be a graph with $V = \{v_1, \ldots, v_n\}$. Then

$$\sum_{i=1}^{n} d(v_i) = 2|E|.$$