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Variables in Mathematics Education

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Abstract. This paper suggests that consistently referring to variables as placeholders is an effective countermeasure for addressing a number of the difficulties students' encounter in learning mathematics. The suggestion is supported by examples discussing ways in which variables are used to express unknown quantities, define functions and express other universal statements, and serve as generic elements in mathematical discourse. In addition, making greater use of the term "dummy variable" and phrasing statements both with and without variables may help students avoid mistakes that result from misinterpreting the scope of a bound variable.

Keywords: variable, bound variable, mathematics education, placeholder.

1 Introduction

Variables are of critical importance in mathematics. For instance, Felix Klein wrote in 1908 that "one may well declare that real mathematics begins with operations with letters,"[3] and Alfred Tarski wrote in 1941 that "the invention of variables constitutes a turning point in the history of mathematics."[5] In 1911, A. N. Whitehead expressly linked the concepts of variables and quantification to their expressions in informal English when he wrote: "The ideas of 'any' and 'some' are introduced to algebra by the use of letters. . . it was not till within the last few years that it has been realized how fundamental *any* and *some* are to the very nature of mathematics."[6] There is a question, however, about how to describe the use of variables in mathematics instruction and even what word to use for them.

Logicians seem generally to agree that variables are best understood as placeholders. For example, Frege wrote in 1893, "The letter 'x' serves only to hold places open for a numeral that is to complete the expression. . . This holding-open is to be understood as follows: all places at which '?' stands must be filled always by the same sign, never by different ones,"[2] and Quine stated in 1950, "The variables remain mere pronouns, for cross-reference; just as 'x' in its recurrences can usually be rendered 'it' in verbal translations, so the distinctive variables 'x','y', 'z', etc., correspond to the distinctive pronouns 'former' and 'latter', or 'first', 'second', and 'third', etc."[4]

The thesis of this article is to suggest that the logicians' view of variables is best for the teaching of mathematics – that, right from the beginning and

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regardless of whether they are called "letters," "literals," "literal symbols," or "variables," they be described as placeholders, and that, to be seen as meaningful, they be presented in full sentences, especially ones with quantification. This thesis will be supported by providing a sampling of the different uses of variables and analyzing the reasons for some of the difficulties students encounter with them. Two that arise repeatedly are (1) thinking of variables as exotic mathematical objects that do not have a clear connection with our everyday universe, and (2) regarding variables as having an independent existence even though they have been introduced as bound by a quantifier.

2 Mathematical Uses of Variables

2.1 Variables Used to Express Unknown Quantities

In the early grades, students are sometimes given problems like the following:

Find a number to place in the box so that $3 + \Box = 10$.

Later, however, when algebra is introduced, the empty-box notation is typically abandoned and the focus shifts to learning rules for manipulating equations in order to get a variable, typically x, on one side and a number on the other. With the resulting emphasis on mechanical procedures, the meaning of "Solve the equation for x" may be obscured, with students coming to view x as a mysterious object with no relation to the world as they know it. Pointing out that x just holds the place for the unknown quantity - perhaps even making occasional use of the empty-box notation even after variables have been introduced - can counteract students' sense that the meaning of x is beyond their understanding.

To solve an equation for x simply means to find all numbers (if any) that can be substituted in place of x so that the left-hand side of the equation will be equal to the right-hand side. In my work with high school mathematics teachers, I have found that a surprising number are unfamiliar with this way of thinking and have never thought of asking their students to test the truth of an equation for a particular value of the variable by substituting the value into the left-hand side and into the right-hand side to see if the results are equal.

By holding the place for the unknown quantity in an equation such as $\sqrt{4-3x} = x$, the variable x enables us to work with it in the same way that we would work with a number whose value we know, and this is what enables us to deduce what its value or values might be. In 1972, the mathematician Jean Dieudonné characterized this approach by writing that when we solve an equation, we operate with "the unknown (or unknowns) as if it were a known quantity... A modern mathematician is so used to this kind of reasoning that his boldness is now barely perceptible to him."[1]

2.2 Variables Used in Functional Relationships

Understanding the use of variables in the definition of functions is critically important for students hoping to carry their study of mathematics to an advanced level. In casual conversation, we might say that as we drive along a route, our distance varies constantly with the time we have traveled. So if we let d represent distance and t represent time, it may seem natural to describe the relationship between t and d by saying that for each change in t there is a corresponding change in d. This language has led many to think of variables such as t and d as objects with the capacity to change. Indeed, the word variable itself suggests such a description.

Addressing this issue, however, Tarski wrote: "As opposed to the constants, the variables do not possess any meaning by themselves... The 'variable number' x could not possibly have any specified property... the properties of such a number would change from case to case... entities of such a kind we do not find in our world at all; their existence would contradict the fundamental laws of thought."[5] Quine expressed a similar caution: "Care must be taken, however, to divorce this traditional word of mathematics [variable] from its archaic connotations. The variable is not best thought of as somehow varying through time, and causing the sentence in which it occurs to vary with it."[4]

We are quick to correct students who write "let a be apples and p be pears," telling them that they should say "let a be the number of apples and p be the number of pears." Similarly, t does not actually represent time but holds a place for substituting the number of hours we have been driving, and d does not actually represent distance but holds a place for substituting the corresponding number of miles traveled during that time. *Thus it is not the t or the d that changes; it is the values (number of hours or number of miles) that may be put in their places.* However, this is a distinction that mathematics teachers rarely emphasize to their students. In fact, mathematicians frequently make statements such as, "As x gets closer and closer to 0, $1-x$ gets closer and closer to 1." This way of describing a variable that represents a numerical quantity may contribute to students' common belief that the number 0.99999... "gets closer and closer to 1 but it never reaches 1."

Even more than in the other areas of mathematics they encounter, students must learn to translate the words we use when we describe a function into language that is meaningful to them. For example, we might refer to "the function $y = 2x + 1$." Taken by itself, however, "$y = 2x + 1$" is meaningless. It is simply a predicate, or open sentence, that only achieves meaning when particular numbers are substituted in place of the variables or when it is part of a longer sentence that includes words such as "for all" or "there exists."

Students need to learn that when we write "the function $y = 2x+1$," we mean "the relationship or mapping defined by corresponding to any given real number the real number obtained by multiplying the given number by 2 and adding 1 to the result." We think of x as holding the place for the number that we start with and y as holding the place for the number that we end up with, and we call x the "independent variable" because we are free to start with any real number whatsoever and y the "dependent variable" because its value depends on the value we start with. Imagining a process of placing successive values into the independent variable and computing the corresponding values to place into the dependent variable can give students a feeling for the dynamism of a functional relationship. However, we need to alert students to the fact that the specific letters used to hold the places for the variables have no meaning in themselves. For example, the given function could just as well be described as "$v = 2u + 1$" or "$q = 2p + 1$," or as "$x \to 2x + 1$" or "$u \to 2u + 1$."

Another way to describe this function is to call it "the function $f(x) = 2x+1$" or, more precisely, "the function f defined by $f(x) = 2x + 1$ for all real numbers x." An advantage of the latter notation is that it leads us to think of the function as an object to which we are currently giving the name f. This notation also makes it natural for us to define "the value of the function f at x" as the number that f associates to the number that is put in place of x. Using the notation $f(x)$ to represent both the function and the value of the function at x, while convenient for certain calculus computations, can be confusing to students.

A variation of the preceding notation defines the function by writing $f(\square) = 2 \cdot \square + 1$, pointing out that for any real number one might put into the box, the value of the function is twice that number plus 1. The empty box representation is especially helpful for work with composite functions. Students asked to find, say, $f(g(x))$ often become confused when both f and g have been defined by formulas that use x as the independent variable. When the functions have been defined using empty boxes the relationships are clearer. For instance, in a calculus class students find it easier to learn to compute $f(x + h)$ if they have previously been shown the definition of f using empty boxes.

2.3 Variables Used to Express Universal Statements

Terms like "for all" and "for some" are called quantifiers because "all" and "some" indicate quantity. In a statement starting "For all x" or "For some x," the "scope of the quantifier" indicates how far into the statement the role played by the variable stays the same, and the variable x is said to be "bound" by the quantifier.

Most mathematical definitions, axioms, and theorems are examples of universal statements, i.e., statements that can be written so as to start with the words "for all." For example, the distributive property for real numbers states that for all real numbers a, b, and c, $ab + ac = a(b + c)$. The variables a, b, and c are bound by the quantifier "for all," and they are placeholders in the sense that no matter what numbers are substituted in their place, the two sides of the equation will be equal. Thus the symbols used to name them are unimportant as long as they are consistent with the original.

In mathematics classes it is common to abbreviate the distributive property (and similar statements) by saying that a certain step of a solution is justified "because $ab + ac = a(b + c)$." However, this usage can lead students to invest a, b, and c with meaning they do not actually have. For instance, some students become confused when asked to apply the distributive property to $cb + ca$ because the a, b, and c are the same symbols used in the statement of the property, and students think of them as continuing to have the same meaning as in the statement, without realizing that the scope of the quantifier extends only to the statement's end.

A different problem arises when the omission of the quantifiers is justified by describing a, b, and c as "general numbers" because this suggests that there is a category of number that lies beyond the ordinary numbers with which students are familiar. For those with a secure sense of the way a, b, and c function as placeholders, this terminology is not misleading, but students with a shakier sense of the meaning of variable may imagine a realm of mysterious mathematical objects whose existence makes them uneasy.

By contrast, if the distributive property is simply described as a template into which any real numbers (or expressions with real number values) may be placed to make a true statement, the mystery disappears and the way is prepared for leading students to an increasingly sophisticated ability to apply the property. Again empty boxes may be helpful. For example, the property can be stated as follows: No matter what real numbers we place in boxes $\square$, $\diamondsuit$, and $\triangle$,

$$\square \cdot \diamondsuit + \square \cdot \triangle = \square \cdot (\diamondsuit + \triangle)$$

Encouraging students to test the template by substituting a variety of different quantities in place of $\square$, $\diamondsuit$, and $\triangle$ provides a gentle introduction both to the logical principle of universal instantiation[1] and to the dynamic aspect of the universal quantifier, and substituting successively more complication expressions into the boxes can develop a sense for the power of the property:

$$\begin{aligned} \mathbf{2}\cdot s + \mathbf{2}\cdot t &= \mathbf{2}\cdot (s+t) \\ 2s + 6 = \mathbf{2}\cdot s + \mathbf{2}\cdot 3 &= \mathbf{2}\cdot (s+3) \\ 2^{100} + 2^{99} = \mathbf{2^{99}}\cdot 2 + \mathbf{2^{99}}\cdot 1 &= \mathbf{2^{99}}\cdot (2+1 \quad [= 2^{99}\cdot 3] \\ \boldsymbol{(x^2-1)}\cdot x + \boldsymbol{(x^2-1)}\cdot (x-3) &= \boldsymbol{(x^2-1)}\cdot (x + (x-3)) \quad [= (x^2-1)(2x-3)] \end{aligned}$$

2.4 Dummy Variables and Questions of Scope

Strictly speaking, the term dummy variable simply refers to any variable bound by a quantifier, but we most often use the term when discussing summations and integrals. For example, given a sequence of real numbers a_0, a_1, a_2, . . and a function f, we make a point of referring to k, i, x, and t as dummy variables to help students understand that

$$\sum_{k=1}^{10} a_k = \sum_{i=1}^{10} a_i \qquad \text{and} \qquad \int_1^2 f(x)dx = \int_1^2 f(t)dt.$$

In fact, it may be helpful to use the term dummy variable whenever we are especially concerned about problems that can result from thinking of variable names as "exceeding their bounds," that is, as having meaning outside the scope

[1] Universal instantiation: If a property is true for all elements of a set, then it is true for each individual element of the set.

determined by their quantification. For instance, it is common to state the definitions of even and odd integers as follows:

For an integer to be even means that it equals $2k$ for some integer k.
For an integer to be odd means that it equals $2k + 1$ for some integer k.

Following such an introduction, many students try to prove that the sum of any even integer and any odd integer is odd by starting their argument as follows:

Suppose m is any even integer and n is any odd integer. Then $m = 2k$ and $n = 2k + 1 \ldots$.

For the definitions of even and odd, however, the binding of each occurrence of k extends only to the end of the definition that contains it. In order to avoid the mistake shown in the example, students must come to understand that the symbol k is just a placeholder, with no independent existence of its own. One way to emphasize this fact is to call k a dummy variable. We can reinforce this characterization by writing each definition several times, using a different symbol for the variable each time. For example we could write the definition of even as:

For an integer to be even means that it equals $2a$ for some integer a.
For an integer to be even means that it equals $2r$ for some integer r.
For an integer to be even means that it equals $2m$ for some integer m.

It is also effective to give an alternative version of the definition that does not use a variable at all:

For an integer to be even means that it equals twice some integer.

In general, asking students to translate between formal statements that contain quantifiers and variables and equivalent informal statements without them is very helpful in developing their ability to work with mathematical ideas.

A few years ago I discovered that when I asked students to write how to read, say, the following expression out loud:

$$\{x \in U \mid x \in A \text{ or } x \in B\}.$$

the most common response was to omit the words "the set of all" and write only "x in U such that x is in A or x is in B." More recently, when teaching about equivalence relations, I learned that part of students' difficulty in interpreting such a set definition was a belief that the variable x had a life outside of the set brackets. When I defined the equivalence class of an element a for an equivalence relation R on a set A as

$$[a] = \{x \in A \mid x \, R \, a\},$$

a number of students had trouble applying the definition, and the question they asked was, "What happened to the x?" However, they were successful after I showed them that the definition could be rewritten with t in place of x and that it could be rephrased without the x as "The equivalence class of a is the set of all elements in A that are related to a."

Instructors who teach students with computer programming experience can draw analogies between the ways variables are used in programs and the ways they are used in mathematics. For example, the name for a "local" variable in a subroutine can be used with a different meaning outside the subroutine, and within the subroutine it can be replaced by any other name as long as the replacement is carried out consistently. This is strikingly similar to the way a mathematical variable acts within a definition or theorem statement.

2.5 Variables Used as Generic Elements in Discussions

A variable is sometimes described as a mathematical "John Doe" in the sense that it is a particular object that shares all the characteristics of every other object of its type but has no additional properties. For example, if we were asked to prove that the square of any odd integer is odd, we might start by saying, "Suppose n is any odd integer." As long as we deduce properties of n^2 without making any assumptions about n other than those satisfied by every odd integer, each statement we make about it will apply equally well to all odd integers. In other words, we could replace n by any odd integer whatsoever, and the entire sequence of deductions about n would lead to a true conclusion. In that sense, n is a placeholder.

To be specific, consider that, by definition, for an integer to be odd means that it equals 2 times some integer plus 1. Because this definition applies to every odd integer, a proof might proceed as follows:

> *Proof*: Suppose n is any odd integer. By definition of odd, there is some integer m so that $n = 2m + 1$. It follows that
>
> $$n^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1.$$
>
> But $2m^2 + 2m$ is an integer, and so n^2 is also equal to 2 times some integer plus 1. Hence n^2 is odd.

Dieudonné's use of the word "boldness" to describe the process of solving an equation by operating on the variable as if it were a known quantity applies equally well to the use of a variable as a generic element in a proof. For instance, by boldly giving the name n to an arbitrarily chosen, but representative, odd integer, we can investigate its properties as if we knew what it was. Then, after we have used the definition of odd to deduce that n equals two times some integer plus 1, we can boldly apply the logical principle of existential instantiation[2] to give that "some integer" the name m in order to work with it also as if we knew what it was.

Occasionally we may be given a problem in a way that asks us to think of a certain variable as generic right from the start. For instance, instead of being asked to prove that the square of any odd integer is odd, we might have been given the problem: "Suppose n is any odd integer. Prove that n^2 is odd." In this case, after reading the first sentence, we should think of n as capable of being

[2] Existential instantiation: If we know or suspect that an object exists, then we may give it a name, as long as we are not using the name for another object in our current discussion.

replaced by any arbitrary odd integer, and we would omit the first sentence of the proof that is given above.

An important use of variables as generic elements in mathematics education occurs in deriving the equations of lines, circles, and other conic sections. For example, to derive the equation of the line through $(3, 1)$ with slope 2, we could start as follows: "Suppose (x, y) is any point on the line." As long as we deduce properties of x and y without making any additional assumptions about their values, everything we conclude about (x, y) will be true no matter what point on the line might be substituted in its place.

We could continue by considering two cases: the first in which $(x, y) \neq (3, 1)$ and the second in which $(x, y) = (3, 1)$. For the first case, we note that what insures the straightness of a straight line is the fact that its slope is the same no matter what two points are used to compute it. Therefore, if the slope is computed using (x, y) and $(3, 1)$, the result must equal 2:

$$\frac{y-1}{x-3} = 2, \quad \text{and so} \quad y - 1 = 2(x - 3). \ (*)$$

This concludes the discussion of the first case. In the second case, $(x, y) = (3, 1)$ and both sides of equation (*) equal zero. So in this case it is also true that $y - 1 = 2(x - 3)$. Therefore, because no assumptions about (x, y) were made except for its being a point on the line, we can conclude that every point (x, y) on the line satisfies the equation $y - 1 = 2(x - 3)$.

3 Conclusion

This paper has advocated placing greater emphasis on the role of variables as placeholders to help address students' difficulties as they make the transition to algebra and more advanced mathematical subjects. Supporting examples were given from a variety of mathematical perspectives. It is hoped that the paper will stimulate additional research to delve more deeply into the issues it raises.

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