Problem 1. (10 points) For each of the following functions, circle what kind of singularity it has at $z = 0$, and compute the residue at $z = 0$.

(a) $f(z) = \dfrac{1}{z\sin(z)}$.

Removable **Pole** **Essential**

(b) $f(z) = \dfrac{1}{\sin(1/z)}$.

Removable **Pole** **Essential**

Solution. (a) $f(z)$ has an **pole** at $z = 0$. To see this, we note that

$$\lim_{z\to 0} z^2 f(z) = \lim_{z\to 0} \frac{z^2}{z\sin(z)} = 1,$$

so in fact, $f(z)$ has a pole of order 2. The residue can be computed using the derivative formula that we often use, or alternatively using the power series for $\sin z$. Thus

$$\begin{aligned}\frac{1}{z\sin(z)} &= \frac{1}{z\left(z - \dfrac{z^3}{6} + \cdots\right)} \\ &= \frac{1}{z^2} \cdot \frac{1}{1 - \dfrac{z^2}{6} + \cdots} \\ &= \frac{1}{z^2} \cdot \left(1 + \frac{z^2}{6} + \cdots\right) \\ &= \frac{1}{z^2} + \frac{1}{6} + \cdots .\end{aligned}$$

Hence

$$\operatorname{Res}\left[\frac{1}{z\sin(z)}, 0\right] = 0.$$

But the easiest way to compute the residue is to note that $f(z) = f(-z)$, i.e., the function $f(z)$ is even, so in its Laurent expansion

$$\sum_{k=-\infty}^{\infty} c_k z^k,$$

we have $c_k = 0$ for every odd k. In particular, we have $c_{-1} = 0$, so the residue is 0.

(b) $f(z)$ has an **essential singularity** at $z = 0$. We can see this by observing that $\lim_{z\to 0} z^m f(z)$ does not exist for any integer $m \geq 0$. To ease notation, let $w = 1/z$, so we're looking at

$$f(w) = \frac{1}{\sin(w)}.$$

If we write $f(w)$ as a Laurent series in w, then the residue is the coefficient of w. Thus

$$\begin{aligned} f(w) &= \frac{1}{\left(w - \frac{w^3}{6} + \cdots\right)} \\ &= \frac{1}{w} \cdot \frac{1}{1 - \frac{w^2}{6} + \cdots} \\ &= \frac{1}{w} \cdot \left(1 + \frac{w^2}{6} + \cdots\right) \\ &= \frac{1}{w} + \frac{w}{6} + \cdots \end{aligned}$$

Thus the Laurent series of $f(z)$ looks like

$$f(z) = z + \frac{1}{6z} + \frac{a}{z^3} + \frac{b}{z^5} + \cdots$$

so

$$\text{Res}\left[\frac{1}{\sin(1/z)}, 0\right] = \frac{1}{6}.$$

Problem 2. (15 points) Compute the values of each of the following integrals.

(a) $\displaystyle\int_{|z|=1} \frac{e^{3z}}{z^3}\,dz.$

(b) $\displaystyle\int_{|z|=1} \frac{1}{z(e^z - 1)}\,dz.$

(c) $\displaystyle\int_{\gamma} \overline{z}\,dz,$ where γ is the line segment from 0 to $1 + i$.

Solution. (a) There is a one pole at $z = 0$, and it is a pole of order 3. The residue theorem gives

$$\begin{aligned}\frac{1}{2\pi i}\int_{|z|=1}\frac{e^{3z}}{z^2}\,dz &= \operatorname{Res}\left[\frac{e^{3z}}{z^2},0\right]\\ &= \operatorname{Res}\left[\frac{1}{z^3}\left(1+3z+\frac{(3z)^2}{2!}+\frac{(3z)^3}{3!}+\cdots\right),0\right]\\ &= \operatorname{Res}\left[\frac{1}{z^3}+\frac{3}{z^2}+\frac{9}{2z}+\frac{9}{2}+\cdots,0\right]\\ &= \frac{9}{2}.\end{aligned}$$

So

$$\int_{|z|=1}\frac{e^{3z}}{z^2}\,dz = 9\pi i.$$

(b) Again there is one pole at $z = 0$, but this time it is a pole of order 2. Letting $f(z) = \frac{1}{z(e^z-1)}$, we have

$$\begin{aligned}\operatorname{Res}\big[f(z),0\big] &= \lim_{z\to 0}\frac{d}{dz}\big(z^2 f(z)\big)\\ &= \lim_{z\to 0}\frac{d}{dz}\left(\frac{z}{e^z-1}\right)\\ &= \lim_{z\to 0}\frac{(e^z-1)-ze^z}{(e^z-1)^2}.\end{aligned}$$

One way to proceed now is to use L'Hopital's rule a couple of times. An easier(?) way is to use the Taylor series expansion of e^z. Thus

$$\begin{aligned}\frac{e^z-1-ze^z}{(e^z-1)^2} &= \frac{\left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots\right)-1-z\left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots\right)}{\left(z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots\right)^2}\\ &= \frac{\left(\frac{z^2}{2}+\frac{z^3}{6}+\cdots\right)-\left(z^2+\frac{z^3}{2}+\frac{z^4}{6}+\cdots\right)}{z^2+z^3+\cdots}\\ &= \frac{-\frac{z^2}{2}+\cdots}{z^2+\cdots}.\end{aligned}$$

So the limit as $z \to 0$ is $-\frac{1}{2}$. Therefore

$$\int_{|z|=1}\frac{1}{z(e^z-1)}\,dz = 2\pi i\operatorname{Res}\left[\frac{1}{z(e^z-1)},0\right] = 2\pi i\cdot\left(-\frac{1}{2}\right) = -\pi i.$$

(c) This line integral needs to be done directly from the definition. The curve γ is parametrized by

$$z(t) = (1+i)t \quad \text{for } 0 \le t \le 1.$$

So

$$\begin{aligned}
\int_\gamma \overline{z}\,dz &= \int_0^1 \overline{(1+i)t}\,d\big((1+i)t\big) \\
&= \int_0^1 (1-i)t\,(1+i)\,dt \\
&= 2\int_0^1 t\,dt \\
&= 2\cdot\frac{1}{2} \\
&= 1.
\end{aligned}$$

Problem 3. (10 points) Use residue theory to compute the value of the definite integral

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx.$$

(I expect you to use complex analysis for this problem, although it can also be done using continued fractions as you learned in first-year calculus.)

Solution. Let

$$f(z) = \frac{z^2}{(z^2+1)^2},$$

and let

$$D_R = \big\{re^{i\theta} : -R < r < R \text{ and } 0 < \theta < \pi\big\}.$$

(This is the usual half-disk in the upper half-plane.) The function $f(z)$ has one pole in D_R, namely $z = i$, and it's a double pole, since

$$f(z) = \frac{z^2}{(z^2+1)^2} = \frac{z^2}{(z+i)^2(z-i)^2}.$$

The residue at $z = i$ is

$$\begin{aligned}
\operatorname{Res}\bigl[f(z), i\bigr] &= \lim_{z\to i} \frac{d}{dz}\bigl((z-i)^2 f(z)\bigr) \\
&= \lim_{z\to i} \frac{d}{dz}\left(\frac{z^2}{(z+i)^2}\right) \\
&= \lim_{z\to i} 2\left(\frac{z}{z+i}\right)\frac{d}{dz}\left(\frac{z}{z+i}\right) \\
&= \lim_{z\to i} 2\left(\frac{z}{z+i}\right)\left(\frac{i}{(z+i)^2}\right) \\
&= 2\cdot\frac{i}{2i}\cdot\frac{i}{-4} \\
&= -\frac{i}{4}.
\end{aligned}$$

So the residue theorem says that

$$\int_{\partial D_R} f(z)\,dz = 2\pi i \operatorname{Res}\bigl[f(z), i\bigr] = \frac{\pi}{2}.$$

Let L_R be the line segment $-R \le x \le R$, and let Γ_R be the semicircle of radius R in the upper halfplane. Then

$$\int_{L_R} f(z)\,dz = \int_{-R}^{R} \frac{x^2}{(x^2+1)^2}\,dx.$$

Next we estimate

$$\begin{aligned}
\left|\int_{\Gamma_R} f(z)\,dz\right| &\le \sup_{z\in\Gamma_R}\bigl|f(z)\bigr|\cdot\text{Length}(\Gamma_R) \quad \text{(ML estimate)}, \\
&= \sup_{z\in\Gamma_R}\left|\frac{z^2}{(z^2+1)^2}\right|\cdot 2\pi R \\
&\le \frac{R^2}{(R^2-1)^2}\cdot 2\pi R \quad \text{(note it's } R^2-1\text{, not } R^2+1\text{,)} \\
&\xrightarrow{R\to\infty} 0.
\end{aligned}$$

So letting $R \to \infty$ and combining these calculations gives

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx = \lim_{R\to\infty}\left(\int_{\partial D_R} f(z)\,dz - \int_{\partial\Gamma_R} f(z)\,dz\right) = \frac{\pi}{2}.$$

Problem 4. (10 points) For each of the following functions, describe the Taylor series expansion about the indicated point, and compute the radius of convergence.

(a) $f(z) = \log(z)$ centered at $z_0 = 2$.

(b) $f(z) = \dfrac{1}{(1-z)^2}$ centered at $z_0 = 0$.

Solution. In general the Taylor series expansion of $f(z)$ centered at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k.$$

Each part of this problem can be done by computing the derivatives at the indicated point. Alternatively, one can use related series and differentiate or integrate them.

(a) We have $f'(z) = \frac{1}{z}$, so

$$f'(z) = \frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2} \cdot \frac{1}{1+\frac{z-2}{2}}.$$

Now we can expand using the geometric series to get

$$f'(z) = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k \left(\frac{z-2}{2}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}}(z-2)^k.$$

Integrating gives

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)2^{k+1}}(z-2)^{k+1} + C.$$

The constant is obtained by setting $z = 2$, so $\log(2) = f(2) = C$. Finally, relabeling, we get

$$f(z) = \log(z) = \log(2) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k2^k}(z-2)^k.$$

The radius of convergence ρ may be computed using the ratio test or the root test. The latter gives

$$\rho^{-1} = \lim_{k\to\infty} \left|\frac{(-1)^{k+1}}{k2^k}\right|^{1/k} = \lim_{k\to\infty} \frac{1}{2k^{1/k}} = \frac{1}{2},$$

so $\rho = 2$.

(b) Again, it's not very hard to compute the derivatives. But even easier to note that $f(z)$ is the derivative of $\frac{1}{1-z}$, which is just a geometric series. So

$$f(z) = \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{d}{dz}\sum_{k=0}^{\infty} z^k = \sum_{k=1}^{\infty} kz^{k-1} = \sum_{k=0}^{\infty}(k-1)z^k.$$

The radius of convergence is

$$\rho = \lim_{k\to\infty} \frac{1}{(k-1)^{1/k}} = 1.$$

Problem 5. (10 points) Let

$$f(z) = \frac{1}{z^2 - 2z}.$$

(a) Find the Laurent series of $f(z)$ centered at 0 in the domain $|z| < 2$.
(b) Find the Laurent series of $f(z)$ centered at 0 in the domain $|z| > 2$.

Solution. (a) We note that $f(z)$ has a pole at $z = 0$, but that's okay. The partial fraction expansion of $f(z)$ is

$$f(z) = \frac{1/2}{z-2} - \frac{1/2}{z}.$$

We leave the second term alone and expand the first using the geometric series

$$\frac{1/2}{z-2} = -\frac{1}{4} \cdot \frac{1}{1-\frac{z}{2}} = -\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} \frac{-1}{2^{n+2}} z^n.$$

This converges on $|z| < 2$. Further, if we also include $n = -1$, we get the other term, so the Laurent series of f on the domain $|z| < 2$ is

$$f(z) = \sum_{n=-1}^{\infty} \frac{-1}{2^{n+2}} z^n.$$

(b) For $|z| > 2$, we want an expansion in the variable $1/z$, so

$$\frac{1/2}{z-2} = \frac{1}{2z} \cdot \frac{1}{1-\frac{2}{z}} = \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \sum_{n=0}^{\infty} \frac{2^{n-1}}{z^{n+1}} = \sum_{n=1}^{\infty} \frac{2^{n-2}}{z^n}.$$

This gives

$$f(z) = -\frac{1}{2z} + \sum_{n=1}^{\infty} \frac{2^{n-2}}{z^n}.$$

We can simplify by noting that the $n = 1$ term cancels the $-1/2z$, so

$$f(z) = \sum_{n=2}^{\infty} \frac{2^{n-2}}{z^n}.$$

Problem 6. (15 points) Let D be a bounded domain with nice boundary.

(a) Suppose that $f(z)$ is analytic on D, continuous on $D \cup \partial D$, and does not vanish on $D \cup \partial D$. Let

$$m = \inf_{z \in \partial D} \bigl| f(z) \bigr|$$

be the *smallest* value of $\bigl|f(z)\bigr|$ on the boundary of D. Prove that

$$\bigl|f(z)\bigr| \geq m \quad \text{for all } z \in D.$$

(This is a *minimum principle* that complements the maximum principle.)

Problem 6. (continued)

(b) Let D be the unit disk. Find a function that is analytic on $D \cup \partial D$ and satisfies $f(0) = 0$, and such that f does *not* satisfy the minimum principle.

(c) Suppose that $f(z)$ is analytic and non-constant on D and continuous on $D \cup \partial D$. Assume further that $\bigl|f(z)\bigr|$ is constant for $z \in \partial D$. Prove that $f(z)$ must have a zero in D.

Solution. (a) For any function h, we write

$$M(h) = \sup_{z \in \partial D} \bigl|h(z)\bigr| \quad \text{and} \quad m(h) = \inf_{z \in \partial D} \bigl|h(z)\bigr|.$$

Since $f(z)$ does not vanish on D, we know that $g(z) = 1/f(z)$ is analytic on D. So we can apply the maximum principle to $g(z)$ to conclude that

$$\bigl|g(z)\bigr| \leq M(g) \quad \text{for all } z \in D.$$

Since $g = 1/f$, this implies that

$$\frac{1}{\bigl|f(z)\bigr|} \leq M(1/f) \quad \text{for all } z \in D.$$

But if T is any set of positive real numbers, we have

$$\sup\left\{\frac{1}{t} : t \in T\right\} = \frac{1}{\inf\{t : t \in T\}}.$$

This implies that $M(1/f) = 1/m(f)$. Substituting this in above gives

$$m(f) \leq \bigl|f(z)\bigr| \quad \text{for all } z \in D.$$

(b) The simplest example is $f(z) = z$. Then $m(f) = 1$, but $\bigl|f(z)\bigr|$ is not larger than $m(f)$. In fact, we have $\bigl|f(z)\bigr| < m(f)$ for all z in the unit circle.

(c) The maximum principle says that

$$\bigl|f(z)\bigr| \leq M(f) \quad \text{for all } z \in D.$$

Suppose that f does not vanish. Then the minimum principle (proven in (a)) says that

$$\bigl|f(z)\bigr| \ge m(f) \quad \text{for all } z \in D.$$

However, we're given that $\bigl|f(z)\bigr|$ is constant for $z \in \partial D$, so $m(f) = M(f)$ directly from the definitions of $m(f)$ and $M(f)$. So our two inequalities imply that

$$\bigl|f(z)\bigr| = M(f) = m(f) \quad \text{for all } z \in D.$$

In particular, there are points $z \in D$ for which $\bigl|f(z)\bigr| = M(f)$, so the other half of the maximum principle tells us that f is constant.

Problem 7. (10 points) Let $f(z)$ be the polynomial

$$f(z) = z^4 + 5z + 1.$$

(a) Prove that $f(z)$ has exactly one root inside the disk $|z| < 1$.
(b) How many roots does $f(z)$ have inside the annulus $1 < |z| < 2$? Prove that your answer is correct.

Solution. (a) For $|z| = 1$ we have

$$|5z| = 5 \ge 2 = |z^4| + 1 \ge |z^4 + 1|.$$

So from Rouché's theorem, the polynomial $f(z)$ and the polynomial $5z$ have the same number of zeros in the disk $|z| < 1$. Since $5z$ clearly has one zero in the disk, so does $f(z)$.
(b) On the circle $|z| = 2$ we have

$$|z^4| = 16 \ge 6 = |5z| + 1 \ge |5z + 1|,$$

so $f(z)$ and z^4 have the same number of zeros in the disk $|z| < 2$. Since z^4 has four zeros (counted with mulitplicity), so does $f(z)$. That's the number of zeros in the disk $|z| < 2$, and we know from (a) that there is one zero in the disk $|z| < 1$, so $f(z)$ has three zeros in the annulus $1 < |z| < 2$.

Problem 8. (10 points) Let $f(z)$ be analytic in a domain D, and suppose that f satisfies

$$\operatorname{Re}\bigl(f(z)\bigr) = \operatorname{Im}\bigl(f(z)\bigr) \quad \text{for all } z \in D.$$

Prove that f is constant in D.

Solution. There are probably lots of ways to do this problem. Here's one. Write $f(z) = u(x, y) + iv(x, y)$ as usual. The assumption that $\mathrm{Re}\big(f(z)\big) = \mathrm{Im}\big(f(z)\big)$ says that

$$u(x, y) = v(x, y).$$

Now the Cauchy–Riemann equations yield

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} && \text{(Cauchy–Riemann equation)} \\ &= \frac{\partial u}{\partial y} && \text{(since } v = u\text{)} \\ &= -\frac{\partial v}{\partial x} && \text{(Cauchy–Riemann equation)} \\ &= -\frac{\partial u}{\partial x} && \text{(since } v = u\text{).} \end{aligned}$$

It follows that

$$\frac{\partial u}{\partial x} = 0.$$

A similar calculation gives

$$\frac{\partial u}{\partial y} = 0.$$

Alternatively, we can use $u_x = 0$ and compute

$$0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y}.$$

Thus $u_x = 0$ and $u_y = 0$, which implies that u is a constant. And since $v = u$, we find that $f = u + iv$ is also a constant.

Problem 9. (10 points) Proe that there exists a function $f(z)$ with the following properties:

- $f(z)$ is meromorphic on $\mathbb{C}$.
- $f(z)$ has simple poles at the points $\{1, 2, 3, 4, \ldots\}$ and no other poles.
- For $k \in \{1, 2, 3, \ldots\}$, the residue of $f(z)$ at k is equal to k.

Be sure to prove that the function that you define is meromorphic, as well as having the indicated poles and residues.

Solution. We'd like to use

$$\sum_{k=1}^{\infty} \frac{k}{z-k},$$

but it doesn't converge. The function $k/(z-k)$ looks like -1 when k is large, so we might try summing

$$\frac{k}{z-k}+1=\frac{z}{z-k}.$$

But $z/(z-k)$ looks like $-z/k$ when k is large, so its sum won't converge, either. So we add on z/k to compensate,

$$\frac{k}{z-k}+1+\frac{z}{k}=\frac{z}{z-k}+\frac{z}{k}=\frac{z^2}{(z-k)k}.$$

Note that this calculation shows that $z^2/((z-k)k$ has a simple pole at $z=k$ with residue k.

Then we define

$$f(z)=\sum_{k=1}^{\infty}\left(\frac{k}{z-k}-1-\frac{z}{k}\right)=\sum_{k=1}^{\infty}\frac{z^2}{(z-k)k}.$$

The usual argument shows that f is meromorphic with the correct poles. We briefly indicate. Choose any R (not an integer) and break up f as

$$f(z)=f_1(z)+f_2(z)=\sum_{k<2R}\frac{z^2}{(z-k)k}+\sum_{k>2R}\frac{z^2}{(z-k)k}.$$

Let $D_R=\{|z|<R\}$ be a disk of radius R. Then $f_1(z)$ is meromorphic on D_R with simples poles at the integers $k<R$ and residue k at k. On the other hand, for $z\in D_R$ and $k>2R$ we have

$$\left|\frac{z^2}{(z-k)k}\right|\le\frac{R^2}{(k-R)k},$$

so

$$\sum_{k>2R}\left|\frac{z^2}{(z-k)k}\right|\le\sum_{k>2R}\frac{R^2}{(k-R)k}<\infty.$$

The Weierstrass M-test implies that the series defining $f_2(z)$ converges to an analytic function on D_R. Hence $f(z)$ is meromorphic on D_R with the desired poles and residues at $\{k<R\}$. Since R is arbitrary, this shows that $f(z)$ is entire with the desired poles and residues.

Problem 10. (10 points) Let $f(z)$ be an analytic function that maps the unit disk conformally to a domain D. In other words, if we denote the unit disk by $\mathbb{D}=\bigl\{|z|<1\bigr\}$, then

$$f:\mathbb{D}\longrightarrow D\quad\text{is analytic, one-to-one, and onto.}$$

Also let

$$m=\inf_{w\in\partial D}\bigl|f(0)-w\bigr|$$

be the distance from $f(0)$ to the boundary of D. Prove that

$$\bigl|f'(0)\bigr| \ge m.$$

(*Hint*. Consider the inverse function $f^{-1} : D \longrightarrow \mathbb{D}$, note that the disk around $f(0)$ of radius m is contained in D, and use Schwarz's lemma.)

Solution. We consider the inverse function

$$f^{-1} : D \longrightarrow \mathbb{D}.$$

Our choice of m tells us that the disk

$$B = \bigl\{w \in \mathbb{C} : \bigl|f(0) - w\bigr| < m\bigr\}$$

is contained in D, so f^{-1} is analytic on B; and since the image of f^{-1} is in $\mathbb{D}$, we know that

$$\bigl|f^{-1}(w)\bigr| \le 1 \quad \text{for all } w \in D.$$

We want to shift B to be the unit disk. The map $z \mapsto mz + f(0)$ send the unit disk to B, so we should look at the function

$$g(z) = f^{-1}\bigl(mz + f(0))\bigr).$$

Then $g : \mathbb{D} \to \mathbb{D}$ with $g(0) = 0$, so the derivative version of Schwarz's lemma says that $\bigl|g'(0)\bigr| \le 1$. Note that

$$g'(0) = (f^{-1})'\bigl(f(0)\bigr)m.$$

So we find that

$$\bigl|(f^{-1})'\bigl(f(0)\bigr)\bigr| \le \frac{1}{m}.$$

Okay, now we differentiate the identity

$$f^{-1}\bigl(f(z)\bigr) = z$$

to get

$$(f^{-1})'\bigl(f(z)\bigr) \cdot f'(z) = 1.$$

Evaluating at $z = 0$ gives

$$(f^{-1})'\bigl(f(0)\bigr) \cdot f'(0) = 1.$$

So

$$(f^{-1})'\bigl(f(0)\bigr) = \frac{1}{f'(0)},$$

and substituting this above gives

$$\left|\frac{1}{f'(0)}\right| \le \frac{1}{m}.$$

Cross-multipying gives

$$m \le \bigl|f'(0)\bigr|,$$

which is the desired result.

Problem 11. (10 points) Compute the value of the integrals

$$\int_0^\infty \cos(x^2)\,dx \quad \text{and} \quad \int_0^\infty \sin(x^2)\,dx$$

Hint #1. Integrate the function $f(z) = e^{iz^2}$ around the boundary of the region

$$D_R = \left\{ re^{i\theta} : 0 < r < R \text{ and } 0 < \theta < \frac{\pi}{4} \right\}.$$

Hint #2. The following integral from 3rd semester calculus may be useful:

$$\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}.$$

Warning. Don't spend too much time on this problem until you've worked on the other problems.

Solution. The function $f(z) = e^{iz^2}$ is entire, so Cauchy's theorem tells us that

$$\int_{\partial D_R} f(z)\,dz = 0.$$

The boundary of D_R consists of 3 pieces:

$$\begin{aligned} L_1 &= \{x : 0 \le x \le R\}, \\ L_2 &= \{t\sqrt{i} : 0 \le t \le R\} \quad \text{(in reverse direction)}, \\ \Gamma_R &= \{Re^{i\theta} : 0 \le \theta \le \pi/4\}. \end{aligned}$$

Here $\sqrt{i}$ is the square root in the first quadrant, i.e., $\sqrt{i} = \frac{1+i}{\sqrt{2}}$.

The integral along L_1 gives the integrals that we're trying to compute,

$$\int_{L_1} f(z)\,dz = \int_0^R e^{ix^2}\,dx = \int_0^R \cos(x^2)\,dx + i\int_0^R \sin(x^2)\,dx.$$

For L_2, we have

$$\begin{aligned} \int_{L_2} f(z)\,dz &= \int_R^0 e^{i(\sqrt{i}t)^2}\,d\left(\sqrt{i}t\right) \\ &= -\sqrt{i}\int_0^R e^{-t^2}\,dt \\ &= -\frac{1+i}{\sqrt{2}}\int_0^R e^{-t^2}\,dt. \end{aligned}$$

Hence

$$\lim_{R\to\infty}\int_{L_2} f(z)\,dz = -\frac{1+i}{\sqrt{2}}\int_0^\infty e^{-t^2}\,dt = -\frac{1+i}{\sqrt{2}}\cdot\frac{\sqrt{\pi}}{2}.$$

Finally, for Γ_R we use Jordan's lemma, which says that if C_R is the semicircle of radius R in the upper halfplane, then

$$\int_{C_R} |e^{iz}|\,|dz| < \pi.$$

Our curve Γ_R is not equal to C_R. By making the change of variables $w = z^4$, we could map Γ_R to C_R, but then we wouldn't get the integral in Jordan's Lemma. So instead we use the change of variables $w = z^2$, which maps Γ_R to the quarter-circle

$$B_R = \{Re^{i\theta} : 0 \le \theta \le \pi/2\}.$$

Then

$$\begin{aligned}\int_{\Gamma_R} f(z)\,dz &= \int_{B_R} f(w^{1/2})\,d(w^{1/2})\\ &= \int_{B_R} e^{iw}\frac{dw}{2w^{1/2}}.\end{aligned}$$

Hence

$$\begin{aligned}\left|\int_{\Gamma_R} f(z)\,dz\right| &= \left|\int_{B_R} e^{iw}\frac{dw}{2w^{1/2}}\right|\\ &\le \int_{B_R} |e^{iw}|\frac{|dw|}{2|w^{1/2}|}\\ &= \frac{1}{2R^{1/2}}\int_{B_R} |e^{iw}||dw|.\end{aligned}$$

In order to use Jordan's lemma, we note that C_R is the union of the quarter-circle B_R and the quarter-circle $B'_R = \{-\overline{z} : z \in B_R\}$. In other words, the map $z \to -\overline{z}$ maps B_R to B'_R. We also note that if $z = x + iy \in B_R$, then $-\overline{z} = -x + iy$, so

$$|e^{-\overline{z}i}| = |e^{(-x+iy)i}| = |e^{-y-ix}| = e^{-y} = |e^{(x+iy)i}| = |e^{zi}|,$$

and similarly

$$|d(-\overline{z})| = |dz|,$$

so

$$\int_{B_R} |e^{iz}|\,|dz| = \int_{B'_R} |e^{iz}|\,|dz|.$$